RA Solutions Wednesday

1) $a_n = \dfrac{2n^2+3}{4n^2+1}$, $L = \dfrac{1}{2}$. Let $\varepsilon > 0$

$$|a_n - L| = \left|\frac{2n^2+3}{4n^2+1} - \frac{1}{2}\right| = \left|\frac{4n^2+6-(4n^2+1)}{2+8n^2}\right|$$

$$= \frac{5}{8n^2+2}$$

Now $8n^2+2 > n$. So $\dfrac{1}{8n^2+2} < \dfrac{1}{n}$

or $\dfrac{5}{8n^2+2} < \dfrac{5}{n}$. Let $N \in \mathbb{N}$, $N > \dfrac{5}{\varepsilon}$.

Then $n \geq N \Rightarrow |a_n - L| = \dfrac{5}{8n^2+2} < \dfrac{5}{n} < 5 \cdot \dfrac{\varepsilon}{5}$

$$= \varepsilon.$$

So $a_n \to \frac{1}{2}$.

(2) We have $a_n = \dfrac{e^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(n+1)!} \Big/ \frac{e^n}{n!} = \frac{e^{n+1}\, n!}{e^n (n+1)!}$$

$$= \frac{e}{n+1} \qquad [(n+1)! = (n+1)\, n!]$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1.$$

So series converges.

3) $f(x) = x^6$. $|f(x) - f(y)| = |x^6 - y^6|$

$$= |x^3 - y^3|\,|x^3 + y^3|$$

$$= |x-y|\,|x^2+xy+y^2|\,|x^3+y^3|$$

$$\leq 6|x-y| \quad \text{on } [0,1]$$

since $|x^2+xy+y^2|\,|x^3+y^3| \leq 6$ on $[0,1]$.

Let $\varepsilon > 0$. Take $\delta = \dfrac{\varepsilon}{6}$. Then $|x-y| < \delta$

$$\Rightarrow |f(x) - f(y)| = |x^6 - y^6| \leq 6|x-y| < 6\,\frac{\varepsilon}{6} = \varepsilon.$$

So f is continuous on $[0,1]$.

(4) $f(x) = \sin(2x) \quad f'(x) = 2\cos(2x)$

For $(x,y) \in [0,\infty)$ $\exists\, c$ such that

$$\frac{f(x)-f(y)}{x-y} = \frac{\sin(2x)-\sin(2y)}{x-y} = f'(c), \quad c \in [x,y]$$

Now $|f'(c)| \le 2$.

So $|\sin(2x) - \sin(2y)| \le 2|x-y|$

(5) $f(x) = \dfrac{1}{(1+x)^2} = -\dfrac{d}{dx}\dfrac{1}{1+x}$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\text{So } -\frac{d}{dx}\left(\frac{1}{1+x}\right) = 1 - 2x + 3x^2 - 4x^3$$

and $$\frac{x^2}{(1+x)^2} = x^2 - 2x^3 + 3x^4 - 4x^5 + \dots$$