Class Test Wednesday Solutions

1) $a_n = \dfrac{n^2-3}{4n^2+1}$. Let $\varepsilon > 0$. Then

$$|a_n - \tfrac{1}{4}| = \left|\frac{n^2-3}{4n^2+1} - \frac{1}{4}\right| = \left|\frac{4n^2-12-(4n^2+1)}{4(4n^2+1)}\right|$$

$$= \frac{13}{4}\,\frac{1}{4n^2+1}.$$

Now $4n^2+1 > n$. So we require

$$\frac{13}{4}\,\frac{1}{4n^2+1} < \frac{13}{4}\,\frac{1}{n} < \varepsilon$$

So $n > \dfrac{13}{4\varepsilon}$. Let $N \in \mathbb{N}$ be such that $N \geq \dfrac{13}{4\varepsilon}$

Then $n \geq N \Rightarrow |a_n - 1/4| < \varepsilon$. So $a_n \to 1/4$

(2) $\dfrac{n}{n^2+2} = n\,\dfrac{1}{n^2+2}$. $n^2+2 > n^2$.

$$\therefore \frac{n}{n^2+2} < \frac{n}{n^2} = \frac{1}{n}.$$

Thus $\displaystyle\sum_{n=1}^{N} \frac{n}{n^2+2} > \sum_{n=1}^{N} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

So $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+2}$ diverges

(3) Let $x, y \in [0,2]$ $f(x) = x^4 + 3x$

$$|f(x) - f(y)| = |x^4 + 3x - y^4 - 3x|$$
$$= |x^4 - y^4 + 3(x-y)|$$
$$\leq |x^4 - y^4| + 3|x-y|$$
$$= |x^2-y^2||x^2+y^2| + 3|x-y|$$
$$= |x-y|\left(|x+y||x^2+y^2| + 3\right)$$
$$\leq 35|x-y| \quad \text{for } x, y \in [0,2]$$

Take $\delta = \frac{1}{35}\varepsilon$. Then $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

So f is continuous

(4) $|e^{\sin x} - e^{\sin y}| \le e|x-y|$ (Note typo on Qu)

Let $\qquad\qquad\qquad\qquad\qquad x, y \in [0, \pi/2]$

$$f(x) = e^{\sin x}$$

$$f'(x) = \cos x\, e^{\sin x}$$

By MVT

$$\frac{e^{\sin x} - e^{\sin y}}{x-y} = f'(c) \qquad c \text{ between } x \text{ and } y$$

$$\therefore |e^{\sin x} - e^{\sin y}| = |f'(c)|\,|x-y|$$

$$= |\cos c \cdot e^{\sin c}|\,|x-y|$$

$$\le e^{\sin c}|x-y|$$

Now $\sin x$ increases on $[0, \pi/2]$

$$\therefore |e^{\sin x} - e^{\sin y}| \le e^{\sin \pi/2}\,|x-y|$$

$$= e|x-y|$$

5) $f(x) = x\sinh(3x)$

$$= x\left(3x + \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \cdots\right)$$

One can obtain this by setting $u = 3x$ in the series for $\sinh u$

$f(u) = \sinh u$

$f'(u) = \cosh u$

$f''(u) = \sinh u$ etc

$\sinh 0 = 0$, $\cosh u = 1$, Result follows from Taylor's formula

Thursday Class Test Solutions RA

1) $a_n = \dfrac{2n^2}{8n^2+1}$. Let $\varepsilon > 0$, $a = 1/4$

$$|a_n - a| = \left|\frac{2n^2}{8n^2+1} - \frac{1}{4}\right| = \left|\frac{8n^2 - 8n^2 - 1}{4(8n^2+1)}\right|$$

$$= \frac{1}{4}\,\frac{1}{8n^2+1}$$

$8n^2+1 > n.$

$$\therefore \frac{1}{4}\,\frac{1}{8n^2+1} < \frac{1}{4}\,\frac{1}{n} < \varepsilon$$

if $n > \dfrac{1}{4\varepsilon}$. Let $N > \dfrac{1}{4\varepsilon}$, $N \in \mathbb{N}$

Then for $n \geq N$ $|a_n - a| < \varepsilon$ so $a_n \to 1/4$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+n^3} < \infty$. To see this notice that $2n+n^3 > n^3$

Thus $\dfrac{n}{2n+n^3} < \dfrac{n}{n^3} = \dfrac{1}{n^2}$. Now

$$\sum_{n=1}^{N} \frac{n}{n^3+2n} < \sum_{n=1}^{N} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+2n} < \infty$

3) $f(x) = 4x^3 + 3x$ $\quad x \in [0,3]$

Let $\varepsilon > 0$

$$|f(x) - f(y)| = |4x^3 - 4y^3 + 3x - 3y|$$

$$\leq 4|x^3 - y^3| + 3|x-y|$$

$$= 4|x-y||x^2+xy+y^2| + 3|x-y|$$

$$\leq |x-y|(4(9+9+9+3))$$

$$= 111|x-y|$$

Let $\delta = \dfrac{\varepsilon}{111}$. Then $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

So f is continuous on $[0,3]$

4) $f(x) = \sin(2x)$. $\quad f'(x) = 2\cos 2x$

MVT says:

$$\frac{f(x)-f(y)}{x-y} = \frac{\sin(2x)-\sin(2y)}{x-y} = f'(c)$$

$$= 2\cos(2c)$$

So

$$|\sin(2x)-\sin(2y)| = 2|\cos(2c)||x-y|$$

$$\leq 2|x-y|$$

(5)

$$\cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + -$$

$$\therefore\ x^3\cosh(2x) = x^3\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} + - \right)$$