RA Class Test 1 Friday 2023 Solutions

1) $a_n = \dfrac{4n^2+6}{5n^2+1}$ $\quad a = \dfrac{4}{5}$. Let $\varepsilon > 0$

$$|a_n - a| = \left|\frac{4n^2+6}{5n^2+1} - \frac{4}{5}\right| = \left|\frac{5(4n^2+6) - 4(5n^2+1)}{5(5n^2+1)}\right|$$

$$= \frac{26}{5}\frac{1}{5n^2+1}$$

$5n^2+1 > n$. So $\dfrac{26}{5}\dfrac{1}{5n^2+1} < \dfrac{26}{5}\dfrac{1}{n} < \varepsilon$

if $n > \dfrac{26}{5\varepsilon}$. Let $N > \dfrac{26}{5\varepsilon}$, $N \in \mathbb{N}$

Then $n \geq N \Rightarrow |a_n - \frac{4}{5}| < \varepsilon$. So $a_n \to 4/5$

2)
$$a_n = \frac{e^{3n}}{(3n)!} \qquad a_{n+1} = \frac{e^{3(n+1)}}{(3(n+1))!}$$

$$= \frac{e^3 e^{3n}}{(3n)!\,(3(n+1))(3n+2)(3n+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{e^3 e^{3n}}{(3(n+1))!} \Big/ \frac{e^{3n}}{(3n)!}$$

$$= \frac{e^3}{3(n+1)(3n+2)(3n+1)} \to 0 < 1.$$

So $\sum_{n=1}^{\infty} a_n$ converges

3) $f(x) = x^4 + 6x^2$. $\quad x \in [0,1]$ Let $\varepsilon > 0$

$$|f(x) - f(y)| = |x^4 - y^4 + 6(x^2 - y^2)|$$
$$\leq 6|x^2 - y^2| + |x^4 - y^4|$$

$$= |x^2 - y^2|\,|x^2 + y^2| + 6|x^2 - y^2|$$
$$= |x - y|\left(|x+y|\,|x^2+y^2| + 6|x+y|\right)$$

$$\leq 16|x - y|. \quad \text{Let } \delta = \varepsilon/16$$

Then $|x-y| < \delta \Rightarrow |f(x) - f(y)| < 16\dfrac{\varepsilon}{16} = \varepsilon$.

So f is continuous on $[0,1]$

(4) $f(x) = \tan(2x)$. $f'(x) = 2\sec^2(2x)$

By MVT $\frac{\tan(2x) - \tan(2y)}{x-y} = 2\sec^2(2c)$

$c \in [0, \pi/8]$

$|2\sec^2(2c)| \leq 4$ for $c \in [0, \pi/8]$

$\therefore |\tan(2x) - \tan(2y)| \leq 4|x-y|$.

(5) Let $f(x) = \ln(1+2x)$

$f(0) = \ln 1 = 0$, $f'(x) = \frac{2}{1+2x}$. $f'(0) = 2$

$f''(x) = \frac{-2^2}{(1+2x)^2}$, $f''(0) = -2^2$, $f'''(x) = \frac{2.2^3}{(1+2x)^3}$

So $f'''(0) = 2.2^3$

$f^{(iv)}(x) = \frac{-2.3.2^4}{(1+2x)^4}$ $f^{(iv)}(0) = -6.2^4$

So

$$\ln(1+2x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots$$

$$= 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots$$

and

$$x\ln(1+2x) = x\left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \cdots\right)$$

Friday test two: Real Analysis Solns
2023

(1) $a_n = \dfrac{3n+6}{7n+11}$. $a = \dfrac{3}{7}$. Let $\varepsilon > 0$

$|a_n - a| = \left| \dfrac{3n+6}{7n+11} - \dfrac{3}{7} \right| = \left| \dfrac{7(3n+6) - 3(7n+11)}{7(7n+11)} \right|$

$= \dfrac{9}{7} \dfrac{1}{7n+11}$

$7n + 11 > n$ So $\dfrac{9}{7}\dfrac{1}{7n+11} < \dfrac{9}{7}\dfrac{1}{n} < \varepsilon$

if $n > \dfrac{9}{7\varepsilon}$. Let $N > \dfrac{9}{7\varepsilon}$, $N \in \mathbb{N}$

Then $n \geq N \Rightarrow |a_n - \frac{3}{7}| < \varepsilon$ So $a_n \to \frac{3}{7}$

(2) $a_n = \dfrac{(2n)!}{(3n)!}$ $a_{n+1} = \dfrac{(2(n+1))!}{(3(n+1))!}$

$\dfrac{a_{n+1}}{a_n} = \dfrac{(2(n+1))!}{(3(n+1))!} \Big/ \dfrac{(2n)!}{(3n)!}$

$= \dfrac{(2n+1)(2n+2)}{(3n+3)(3n+2)(3n+1)} \to 0 < 1$

So $\sum a_n$ converges by ratio test

3) $f(x) = x^3 + 4x^2$, Let $\varepsilon > 0$ $x \in [0,2]$

$|f(x) - f(y)| = |x^3 - y^3 + 4x^2 - 4y^2|$

$= |x^3 - y^3 + 4(x-y)(x+y)|$

$\leq |x-y| \, |4(x+y) + x^2 + xy + y^2|$

$\leq 28|x-y|$. Let $\delta = \varepsilon/28$

Then $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

So f is continuous on $[0,2]$

(4) $f(x) = \cos(x^2)$, $\quad x \in [0, \sqrt{\frac{\pi}{2}}]$

$f'(x) = -2x\sin(x^2)$

By MVT

$$\frac{f(x) - f(y)}{x - y} = f'(c) \quad , \quad c \in [0, \sqrt{\tfrac{\pi}{2}}]$$

So $|\cos x^2 - \cos x^2| = |f'(c)|\,|x-y|$

$$\leq 2\sqrt{\frac{\pi}{2}} \sin\frac{\pi}{2}\,|x-y|$$

$$= \sqrt{2\pi}\,|x-y|$$

(5) $\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots$

$$x^3\cos(2x) = x^3\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + - \right)$$