Question 1 (2 + 7 + 5 + 6 = 20 marks)

(a) State the Least Upper Bound axiom.

(b) State the formal $\epsilon - N$ definition for the convergence of a sequence. By using this formal definition of convergence of a sequence, show that the sequence $\left\{\dfrac{2n^2+1}{3n^2+2}\right\}$ converges to $2/3$. It is NOT sufficient to use arithmetic of limits theorem.

(c) Show that if a sequence $\{a_n\}$ is convergent then there exists a positive constant $M < \infty$ such that for each $n \in \mathbb{N}$, $|a_n| < M$.

(d) (i) State the definition of a Cauchy sequence.

(ii) Show that the sequence $x_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$ satisfies the condition

$$|x_{n+1} - x_n| < \frac{1}{n+1},$$

but the sequence $\{x_n\}$ is not Cauchy.

OVER....

Question 2 (8 + 5 + 3 + 4 = 20 marks)

(a) Classify the following statements as true or false. If true, briefly explain why. Stating a relevant theorem is sufficient. If false, explain why not or provide a counterexample.

(i) If the absolute value $|f|$ of a function f is continuous on $(0,1)$, then f must be continuous.

(ii) Every continuous function on a compact interval is uniformly continuous.

(iii) If f is bounded on (a,b) then f is bounded on $[a,b]$ and attains its maximum and minimum values on $[a,b]$.

(iv) If f and g are uniformly continuous on $X \subseteq \mathbb{R}$, then fg is uniformly continuous on X.

(b) Using the $\epsilon - \delta$ definition of continuity, prove that the function $f(x) = \dfrac{x}{1+x}$ is continuous at $x = 1$.

(c) Consider the sequence with nth term $x_n = \sin(n)$. Explain why $\{x_n\}$ has a convergent subsequence.

(d) Let f be a continuous function on an interval I. Suppose that there is a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in I$. Let $\{a_n\}$ be a Cauchy sequence with $a_n \in I$ for each n. Prove that $\{f(a_n)\}$ is also a Cauchy sequence.

Question 3 (6 + 10 + 4 = 20 marks)

(a) (i) Use the ϵ-δ definition for limits to show that

$$\lim_{x\to 1}\frac{2x^2}{3x^2+7}=\frac{1}{5}.$$

(ii) State Rolle's Theorem.

(b) (i) State the Mean Value Theorem. Use it to prove that if $f'(x)=0$ for every $x\in(a,b)$, then f is constant on (a,b).

(ii) Show that the function

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}} & x\neq 0\\ 0 & x=0\end{cases}$$

is differentiable at $x=0$ and $f'(0)=0$.

(c) Prove that if f and g are differentiable at x_0 then fg is differentiable at x_0 and

$$(fg)'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0).$$

Question 4 (6 + 4 + 6 + 4= 20 marks)

(a) Test the following series for convergence.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{e^n}{(2n)!}, \qquad \text{(ii)} \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}.$$

(b) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series. Show that

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

(c) (i) State the Weierstrass M-Test.

(ii) Show that the series of functions

$$\sum_{n=1}^{\infty} \frac{\cos(2n+1)x}{(n+1)^2}$$

is uniformly convergent on $\mathbb{R}$.

(d) Show that the sequence of functions $f_n(x) = \frac{x^3\cos(nx)}{x^2+n^2}$ converges uniformly on $[0, 1]$. What is the limit?

Question 5 (5 + 7 + 3 + 5 = 20 marks)

(a) (i) Define the radius of convergence of a power series.

(ii) Find the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{n}{2^n} x^{2n+2}.$$

(b) (i) State Taylor's Theorem.

(ii) Observe that $\tan^{-1} x = \displaystyle\int_0^x \frac{dt}{1+t^2}$. By computing the Taylor series expansion of $f(t) = \dfrac{1}{1+t^2}$, establish Gregory's series for π. That is, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

(c) Suppose that the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} a_n x^n$ is R_1 and the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} b_n x^n$ is R_2. What is the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) x^n$? Briefly justify your answer.

(d) Let f be at least 3 times continuously differentiable on (a, b). Let $x_0 \in (a, b)$. By considering the Taylor expansion of f show that

$$\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0).$$

(Hint: Using the substitution $x - x_0 = h$ in the usual Taylor expansion show that for some $c \in (x_0, x_0 + h)$

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2} h^2 + \frac{f'''(c)}{3!} h^3.$$

Then use this to find an expression for $f(x_0 - h)$. Then add these together.)

OVER....

Question 6 (8 + 6 + 6= 20 marks)

(a) Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a bounded function.

(i) Let P be a partition of $[a, b]$. Define the upper and lower sums of f on P.

(ii) Define the upper and lower integrals of f on P.

(iii) Define: f is Riemann integrable on $[a, b]$.

(iv) State Riemann's condition for a function to be integrable.

(b) Let f be a continuous function on $[a, b]$ which is non-negative. Suppose that $\int_a^b f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

(c) Show that if $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ which converges uniformly to f on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$