Sample Test Solutions

1) (a) $a_n = \frac{n^2}{2n^2+1}$, $L = \frac{1}{2}$. Let $\varepsilon > 0$

$$|a_n - L| = \left|\frac{n^2}{2n^2+1} - \frac{1}{2}\right| = \left|\frac{2n^2 - 2n^2 - 1}{4n^2+2}\right|$$

$$= \left|\frac{1}{4n^2+2}\right| < \frac{1}{n^2}$$

$\left(4n^2+2 > n^2 \Rightarrow \frac{1}{4n^2+2} < \frac{1}{n^2}\right)$

So if $N \in \mathbb{N}$ and $N \geq \frac{1}{\sqrt{\varepsilon}}$

then $n \geq N \Rightarrow |a_n - L| < \frac{1}{N^2} < \varepsilon$.

$\therefore a_n \to \frac{1}{2}$.

(b). $a_n = \frac{n}{1+n} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n}$. $L = 1$

Let $\varepsilon > 0$

$|a_n - L| = \left|1 - \frac{1}{n} - 1\right| = \frac{1}{n}$. If $n \geq N \geq \frac{1}{\varepsilon}$

then $|a_n - 1| < \varepsilon$. So $a_n \to 1$

(2) $a_n = \frac{e^n}{(n!)^n}$. $a_n^{1/n} = \frac{e}{n!} \to 0$ as $n \to \infty$.
< 1.

So by nth root test

$$\sum_{n=1}^{\infty} a_n < \infty$$

(3) $a_n = n\left(\sqrt{n^2+n+2} - \sqrt{n^2+n}\right)$

$$= n\left(\sqrt{n^2+n+2} - \sqrt{n^2+n}\right)\frac{\left(\sqrt{n^2+n+2} + \sqrt{n^2+n}\right)}{\left(\sqrt{n^2+n+2} + \sqrt{n^2+n}\right)}$$

$$= \frac{n\left(n^2+n+2 - (n^2+n)\right)}{\sqrt{n^2+n+2} + \sqrt{n^2+n}}$$

$$= \frac{2n}{\sqrt{n^2+n+2} + \sqrt{n^2+n}} = \frac{2n}{n\left(\sqrt{1+\frac{1}{n}+\frac{2}{n^2}} + \sqrt{1+\frac{1}{n}}\right)}$$

$$= \frac{2}{\sqrt{1+\frac{1}{n}+\frac{2}{n^2}} + \sqrt{1+1/n}}$$

So $$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} 2}{\lim_{n\to\infty}\left(\sqrt{1+\frac{1}{n}+\frac{2}{n^2}} + \sqrt{1+\frac{1}{n}}\right)}$$

$$= \frac{2}{1+1} = 1.$$

(ii) $$a_n = \frac{2^n+n^3}{3^n+n^2} = \frac{2^n\left(1+\frac{n^3}{2^n}\right)}{3^n\left(1+\frac{n^2}{3^n}\right)}.$$

Now $\lim_{n\to\infty} \frac{n^3}{2^n} = 0$, $\lim_{n\to\infty} \frac{n^2}{3^n} = 0$ (Why?)

So $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n \frac{\lim_{n\to\infty} 1+\frac{n^3}{2^n}}{\lim_{n\to\infty} 1+\frac{n^2}{3^n}}$$

$$= 0.$$

(4) $|S_n| \le M$ all n, where $M>0$ and $M<\infty$

Let $\varepsilon>0$. Choose $N\in\mathbb{N}$ such that $n\ge N \Rightarrow |t_n| < \frac{\varepsilon}{M}$.

Then all $n\ge N$ $$|S_n t_n - 0| = |S_n t_n| \le M|t_n|$$

$$< M\frac{\varepsilon}{M} = \varepsilon$$

So $S_n t_n \to 0$.

(5) $2\sqrt{n}+3\sqrt{n+1} < 2\sqrt{n+1}+3\sqrt{n+1}$
$= 5\sqrt{n+1}$

So $\dfrac{1}{2\sqrt{n}+3\sqrt{n+1}} > \dfrac{1}{5\sqrt{n+1}}$

$\therefore \displaystyle\sum_{n=1}^{N} \frac{1}{2\sqrt{n}+3\sqrt{n+1}} > \sum_{n=1}^{N} \frac{1}{5\sqrt{n+1}} \to \infty$ as $N \to \infty$.

Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+3\sqrt{n+1}}$ is divergent

(6) $x^3-y^3 = (x-y)(x^2+yx+y^2)$
Now on $[0,5]$ $|x^2+xy+y^2| \le 75$.
Let $\varepsilon > 0$. Let $\delta = \dfrac{\varepsilon}{75}$

Then $|x-y| < \delta \Rightarrow |f(x)-f(y)|$
$= |x^3-y^3|$
$= |x-y||x^2+xy+y^2|$

$\le 75|x-y| < 75\dfrac{\varepsilon}{75} = \varepsilon$

So f is continuous on $[0,5]$

(7) Given $x, y \in \mathbb{R}$ close, we can find n such that
$x+nk, y+nk \in [-2k, 2k]$, k period of f
Now f is uniformly continuous on $[-2k,2k] = A$
So if $\bar{x}, \bar{y} \in A$ and close, $f(\bar{x}), f(\bar{y})$ are close.
More precisely we can find $\delta > 0$ such that $|\bar{x}-\bar{y}| < \delta \Rightarrow |f(\bar{x})-f(\bar{y})| < \varepsilon$
Let $\bar{x} = x+nk$, $y+nk = \bar{y}$
Then $|x+nk-(y+nk)| = |x-y| < \delta$
$\Rightarrow |f(x+nk)-f(y+nk)|$
$= |f(x)-f(y)| < \delta$ by periodicity
$\therefore f$ is uniformly continuous on $\mathbb{R}$.

(8) By Taylor's Theorem

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$
$$+ \frac{1}{3!}f'''(\xi)(x-a)^3$$

If x is close to a we can neglect

Now if $f'(a) = 0$ we have an extreme point. $(x-a)^2 > 0$

So if $f(a) > f(x)$

$$f(x) - f(a) < 0 = \frac{1}{2}f''(a)(x-a)^2 + \text{small term}$$

$$\therefore f''(a) < 0.$$

(9) $f(x) = x^3 + 3x + 2$. This is increasing on $[0, \infty)$, so f^{-1} exists on $[0, \infty)$

By the inverse function theorem

$$(f^{-1})'(2) = \frac{1}{f'(2)}$$
$$= \frac{1}{3(2^2)+3} = \frac{1}{15}$$

(10) $f(x) = \cosh x$ $\quad f(0) = 1,\ f'(0) = 0,\ f''(0) = 1$ etc

So

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

(compare with $\cos x$).