Sample Test Two Solutions

(1) $a_n = \dfrac{n^2+4}{3n^2+6} \to \dfrac{1}{3}$.

Let $\varepsilon > 0$. Then $|a_n - L| = \left|\dfrac{n^2+4}{3n^2+6} - \dfrac{1}{3}\right|$

$$= \left|\frac{3(n^2+4) - (3n^2+6)}{3(3n^2+6)}\right| = \frac{6}{3(3n^2+6)}$$

$$= \frac{2}{3n^2+6}.$$

Now $3n^2+6 > n$. So $\dfrac{2}{3n^2+6} < \dfrac{2}{n}$.

$\therefore$ If $\dfrac{2}{n} < \varepsilon$ (or $n > \frac{2}{\varepsilon}$) we have convergence. Let $N \in \mathbb{N}$, $N \geq \frac{2}{\varepsilon}$. Then $n \geq N \Rightarrow |a_n - L| < \varepsilon$. So $a_n \to L$.

(2)(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+4}$. $\quad n + \frac{4}{n} < 4n$, since $\frac{4}{n} < n$ $(n > 4)$

So $\dfrac{n}{n^2+4} > \dfrac{4}{n}$, $n > 4$

Thus $$\sum_{n=1}^{N} \frac{n}{n^2+4} = \left(\sum_{n=1}^{4} + \sum_{n=5}^{N}\right)\frac{n}{n^2+4}$$

$$> \sum_{n=1}^{4} \frac{n}{n^2+4} + \sum_{n=1}^{N} \frac{4}{n} \to \infty$$

as $N \to \infty$. So series diverges.

(b) $\dfrac{n!}{((n+1)!)^2} = \dfrac{n!}{(n+1)!\,(n+1)\,n!} = \dfrac{1}{(n+1)!\,(n+1)} = a_n$

By ratio or comparison test, series converges $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)!\,(n+1)}{(n+2)!\,(n+2)} = \dfrac{(n+1)}{(n+2)^2} \to 0 < 1$

as $n \to \infty$

(c) $n^2 + \ln n > n^2$. $\therefore \dfrac{1}{n^2+\ln n} < \dfrac{1}{n^2}$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

So series converges by comparison test.

Let $\varepsilon > 0$

(3) $f(x) = x^4$. $|f(x) - f(y)| = |(x^2-y^2)(x^2+y^2)|$
$= |x-y||x+y||x^2+y^2|$
$\leq |x-y||10+10||10^2+10^2|$
$= 4000|x-y|$.

Let $\delta = \frac{\varepsilon}{4000}$. $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < |x-y| 4000 \frac{\varepsilon}{4000}$

$= \varepsilon$

So f is continuous.

(4)(a) $f(x) = \cos(2x)$. $\cos\pi = -1$.
$f'(x) = -2\sin(2x)$. $f'(\frac{\pi}{2}) = -2\sin(\pi) = 0$
$f''(x) = -2^2\cos(2x)$
$f''(\frac{\pi}{2}) = -4\cos(\pi) = 4$
$f'''(x) = +2^3\sin(2x)$ $f'''(\frac{\pi}{2}) = 2^3\sin\pi = 0$
$f^{iv}(x) = 2^4\cos(2x)$. $f^{(iv)}(\frac{\pi}{2}) = 2^4$

So $f(x) = -1 - \frac{2^2(x-\frac{\pi}{2})^2}{2!} + \frac{2^4(x-\frac{\pi}{2})^4}{4!} - \frac{2^6(x-\frac{\pi}{2})^6}{6!}$
$+ \ldots$

(b) $\tan^{-1}u = \int_0^u \frac{dt}{1+t^2} = u - \frac{u^3}{3} + \frac{u^5}{5} - \ldots$

So $\tan^{-1}x^2 = x^2 - \frac{x^6}{3} + \frac{x^{15}}{5} \ldots$

(5) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \leq 1$. Let $f(x) = \tan^{-1}x$

So by MVT $\frac{f(x)-f(y)}{x-y} = \frac{1}{1+\xi^2}$ some ξ

or $|\tan^{-1}x - \tan^{-1}y| \leq |x-y|$