The volume of a cone, without calculus

The volume V of a cone with base area A and height h is well known to be given by $V = \frac{1}{3}Ah$. The factor $\frac{1}{3}$ arises from the integration of x^2 with respect to x. The object of this note is to start by supposing $V = cAh$, and to show–*without calculus*–that $c = \frac{1}{3}$. Using the cone formula, we'll also deduce the volume and the surface area of a sphere of radius R.

Consider the frustum of height h, top area a, and base area A, cut from a cone of height $e + h$ (e is for "extra") and base area A. The volume of the frustum is

$$V = cA(e+h) - cae.$$

Now, the area of a cross-section of the cone is proportional to the square of its distance from the vertex, so

$$\frac{\sqrt{a}}{e} = \frac{\sqrt{A}}{e+h}.$$

It follows that

$$e = \frac{\sqrt{a}}{\sqrt{A}-\sqrt{a}}h, \quad e+h = \frac{\sqrt{A}}{\sqrt{A}-\sqrt{a}}h$$

and the volume of the frustum is

$$\begin{aligned} V &= cA\left(\frac{\sqrt{A}}{\sqrt{A}-\sqrt{a}}h\right) - ca\left(\frac{\sqrt{a}}{\sqrt{A}-\sqrt{a}}h\right) \\ &= c\left(\frac{A\sqrt{A}-a\sqrt{a}}{\sqrt{A}-\sqrt{a}}\right)h = c(A+\sqrt{Aa}+a)h. \end{aligned}$$

Now consider what happens as a tends to A. The frustum becomes a cylinder, and we find that $V = 3cAh$. But we know that, for a cylinder, $V = Ah$, so $c = \frac{1}{3}$, and we conclude that the volume of a cone is

$$V = \frac{1}{3}Ah.$$

As a bonus, we obtain the volume of a frustum:

$$V = \frac{1}{3}(A + \sqrt{Aa} + a)h.$$

We conclude with two simple applications of the formula.

The volume of a sphere FIGURE 1 shows a sphere radius R, together with a cylinder of radius R and length $2R$; cones are drilled out from each end of the cylinder to its center.

If we slice each object at a distance x from its center, the area of the slice is, in each case, $\pi(R^2 - x^2)$. Thus the two solids have the same volume, and we conclude that

$$V = \pi R^2 \cdot 2R - 2 \cdot \frac{1}{3} \cdot \pi R^2 \cdot R = \frac{4}{3}\pi R^3.$$

The surface area of a sphere Given a sphere, we divide the surface into very many small (flat) pieces of area A_i, $i = 1, \cdots, n$. We join each to the center, forming sharp cones.

The volume of a typical cone is $V = \frac{1}{3}A_i R$, and the total volume of all the cones is

$$V = \frac{1}{3}R\sum_{i=1}^{n} A_i = \frac{1}{3}RS,$$

where S is the surface area of the sphere. Thus $\frac{1}{3}RS = \frac{4}{3}\pi R^3$, and so

$$S = 4\pi R^2.$$