Real Analysis. The Fundamental Theorem of Calculus says:

Let $f:[a,b] \to \mathbb{R}$ be continuous. Then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This implies that if $F' = f$

then $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Q: Why is this true? In this subject you will find out.

Least upper bound axiom

An upper bound for a set of real numbers S is a real number u such that if $x \in S$ then $x \leq u$.
A lower bound l for S has the property that $x \geq l$ all $x \in S$.

Example $S = [0,1]$.
$u = 1$ is an upper bound.
$2, 17, \pi \ldots$ are upper bounds.
$l = 0$ is a lower bound
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The least upper bound is a number $\bar{u}$ such that

$$x \leq \bar{u} \quad \text{all } x \in S$$

and if w is another upper bound then $w \geq \bar{u}$.
This number is called the supremum of S or $\sup S$.

The greatest lower bound of S is a number $\bar{l}$ such that

$$x \geq \bar{l} \quad \text{all } x \in S$$

and if z is a lower bound, then

$$\bar{l} \geq z$$

$\bar{l}$ is called the infimum or inf.

<u>Axiom</u> Every non empty set of real numbers which is bounded above has a least upper bound.

<u>Theorem 1.13</u> A non empty set of real numbers bounded below has a greatest lower bound.

<u>Proof</u> Suppose that A is non empty and bounded below. Then the set

$$-A = \{-x : x \in A\}.$$

This set is non empty and bounded above. Since if l is l.b for A, $-l$ is an u.b for $-A$. So $-A$ has a least upper bound u. Then $-u$ is the greatest lower bound for A.

<u>Exercise</u> — Justify this last claim

Sets of numbers

<u>Example</u> $S = \mathbb{Q}$ — Rational numbers

$\mathbb{N}$ — Natural numbers $\{1, 2, 3, \ldots\}$

$\mathbb{Z}$ — integers

$\mathbb{R}$ — real numbers

$\mathbb{C}$ — complex numbers

$\mathbb{H}$ — Quarternions

$\mathbb{O}$ — Octonians

Aside

$i^2 = j^2 = k^2 = -1$

$ij = k \ldots$

$a + \hat{i}b + \hat{j}c + \hat{k}d$

$a(bc) = (ab)c$

Proposition $\sqrt{2}$ is irrational

Proof Let $a, b \in \mathbb{N}$, and they have no common factors. ~~Let~~ And

$$\sqrt{2} = \frac{a}{b}.$$

So $2 = a^2/b^2 \Rightarrow a^2 = 2b^2$

$\therefore a^2$ is even $\Rightarrow$ a is even.

($a = 2k+1$ $\quad a^2 = 4k^2 + 4k + 1$ odd)

$a = 2k$ $\quad a^2 = 4k^2$ even)

$\therefore$ ~~$\Rightarrow$~~ $a = 2k$

$a^2 = 4k^2 = 2b^2$

$\therefore b^2 = 2k^2$ $\therefore$ b is even.

Contradiction, since a, b have no common factors. $\therefore$ $\sqrt{2}$ is irrational

Statement Every natural number is either a perfect square or has irrational square root. Can you prove this?

Let $S = \{x \in \mathbb{Q} \quad x^2 < 2\}$

$\sup S = \sqrt{2}$. But $\sqrt{2} \notin S$.

$\sup [0,1) = 1$. But $1 \notin S$.

The supremum of a set does not have to be in S.

Homework Read Theorem 1.14

Why is this important?

Sequences

The Archmidean Property. There is no largest integer. So given x, y we can find $n \in \mathbb{Z}$ such that

$$nx > y.$$

The absolute value of a real number a is $|a| = \sqrt{a^2}$.
$ab \leq |ab| \leq |a||b|$ (Exercise)

A sequence in $\mathbb{R}$ is a collection $\{a_n\}_{n=1}^{\infty}$, where each a_n is a real number.

Example (1) $a_n = \frac{1}{n}$
This is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

(2) $a_n = \left(1 + \frac{1}{n}\right)^n$
$a_1 = 2$, $a_2 = \left(1+\frac{1}{2}\right)^2 = \frac{9}{4}$, $a_3 = \left(\frac{4}{3}\right)^3$
[in fact $\lim_{n\to\infty} a_n \doteq e$.]

(3) $a_n = \frac{n}{n^2+1}$, $a_1 = \frac{1}{2}$, $a_2 = \frac{2}{5}, \ldots$

Definition
A sequence a_n is said to converge to a if for any $\varepsilon > 0$, there exists a natural number N such that if $n \geq N$

$$|a_n - a| < \varepsilon. \qquad (*)$$

If $\{a_n\}$ does not converge, it diverges

[eg. $a_n = (-1)^n$, $-1, 1, -1, 1, -1$.
diverges]
We say $\lim_{n\to\infty} a_n = a$, if $(*)$ holds.
We require now the triangle inequality

Theorem
If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$.
This is the triangle inequality
Proof

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$\leq |a|^2 + 2|a||b| + |b|^2$$
$$= (|a| + |b|)^2$$

Take $\sqrt{\ }$ $\sqrt{(a+b)^2} = |a+b| \leq |a| + |b|$.

Theorem A convergent sequence has only one limit.
Proof Let x_n be a sequence and say $x_n \to x$. Now suppose $x_n \to y$.
Consider
$$|x-y| = |x - x_n + x_n - y|$$
$$\leq |x - x_n| + |x_n - y|$$
$$= |x_n - x| + |x_n - y|$$

We know $x_n \to x$. So given any $\varepsilon > 0$, we can find N_1 such that
$$n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$$
Choose $N_2 \Rightarrow |x_n - y| < \varepsilon/2$

Then $N = \max\{N_1, N_2\}$
Then $n \geq N \Rightarrow |x-y| \leq |x - x_n| + |x_n - y|$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

This is true for any $\varepsilon > 0$. This means $|x-y| = 0$. $\therefore x = y$.
So limit is unique.

Example Let $a_n = \frac{1}{n}$ Show $a_n \to 0$
Let $\varepsilon > 0$. Then $|a_n - 0| = \frac{1}{n} < \varepsilon$
$\Rightarrow n > 1/\varepsilon$ Let $N \in \mathbb{N}$, $N > 1/\varepsilon$
Then $n \geq N \Rightarrow |\frac{1}{n} - 0| < \varepsilon$. So $\frac{1}{n} \to 0$

Example $a_n = \frac{n^2}{n^2+4} = \frac{\cancel{n^2}}{\cancel{n^2}}\left(\frac{1}{1+4/n^2}\right) \Rightarrow 1$

Let $\varepsilon > 0$. Then $\left|\frac{n^2}{n^2+4} - 1\right|$

$$= \left|\frac{n^2 - n^2 - 4}{n^2+4}\right| = 4\left|\frac{1}{n^2+4}\right|$$

$n^2+4 > n \quad \therefore \frac{1}{n^2+4} < \frac{1}{n}$.

So $4\left|\frac{1}{n^2+4}\right| < \frac{4}{n}$.

If $\frac{4}{n} < \varepsilon$, then $\left|\frac{n^2}{n^2+4} - 1\right| < \varepsilon$

Let $N > \frac{4}{\varepsilon}$ and $N \in \mathbb{N}$

Then $n \geq N \Rightarrow \left|\frac{n^2}{n^2+4} - 1\right| \leq \frac{4}{n} < \varepsilon$.

So $\frac{n^2}{n^2+4} \rightarrow 1$.