Real Analysis II Workshop 2

We have not solved all our problems. We were greatly confused about many things. However now we are confused about more important things and at a higher level.

A sequence $\{x_n\}$ is bounded if $\exists M>0$, $M<\infty$, $|x_n| \le M$ all n.

Sequences continued.

Theorem

Every convergent sequence is bounded.

Proof If $x_n \to x$ (x_n converges to x) then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that ~~for~~ $n \ge N$ $|x_n - x| < \varepsilon$.

Take $\varepsilon = 1$. There is an N such that

$$n \ge N \Rightarrow |x_n - x| < 1.$$

Let $M = \operatorname{Max}\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1+|x|\}$

Clearly if $1 \le n \le N-1$, $|x_n| \le M$

If $n \ge N$

$$|x_n| = |x_n - x + x| \le |x_n - x| + |x|$$
$$< 1 + |x| \le M.$$

Thus the sequence is bounded.

Sequences are often iterative.

Example If we want to solve $f(x) = 0$

One method is Newton's Method. Guess solution x_1. Then generate sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

[Converges quadratically]

Example $x_{n+1} = k x_n$, $n \ge 1$.

$x_2 = k x_1$, $x_3 = k x_2 = k^2 x_1$,

$x_4 = k x_3 = k^3 x_1$. $x_n = k^{n-1} x_1$

The convergence depends on k.
If $|k|>1$, sequence diverges
$k=1$, $x_n \to x_1$, $|k|<1$, $x_n \to 0$.
If $k=-1$, sequence diverges.

Theorem 1.19 Let a b be constants and suppose $\{x_n\}$, $\{y_n\}$ converge to x and y respectively. Then

(1) $ax_n \to ax$

(2) $ax_n+by_n \to ax+by$

(3) $x_ny_n \to xy$.

(4) If y_n is never zero $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$.

Proof (1) If $x_n \to x$ then given $\varepsilon>0$ we can find N, such that $n \geq N \Rightarrow |x_n-x|<\varepsilon$.

Assume $a \neq 0$ ($a=0$ is trivial)

$$|ax_n-ax| = |a||x_n-x|.$$

Choose N_1, s.t. $n \geq N_1 \Rightarrow |x_n-x| < \frac{\varepsilon}{|a|}$.

Then $n \geq N_1$, $|ax_n-ax| = |a||x_n-x|$

$$< |a|\frac{\varepsilon}{|a|} = \varepsilon$$

(2) $|ax_n+by_n-ax-by| = |a(x_n-x)+b(y_n-y)|$

($a,b \neq 0$)

Let $\varepsilon>0$ $\leq |a||x_n-x|+|b||y_n-y|$

Let N_1 be such that $n \geq N_1 \Rightarrow |x_n-x| < \frac{\varepsilon}{2|a|}$

Let N_2 be such that $n \geq N_2 \Rightarrow |y_n-y| < \frac{\varepsilon}{2|b|}$.

Let $N = \max\{N_1, N_2\}$

Then if $n \geq N$

$$|ax_n+by_n-ax-by| \leq |a||x_n-x|+|b||y_n-y|$$

$$< |a|\frac{\varepsilon}{2|a|} + |b|\frac{\varepsilon}{2|b|}$$

$$= \varepsilon$$

So $ax_n+by_n \to ax+by$

(3) $x_n y_n \to xy$. $|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy|$

$$= |(x_n - x) y_n + x(y_n - y)|$$

$$\leq |y_n(x_n - x)| + |x(y_n - y)|$$

$y_n \to y$, so y_n is bounded. Thus $\exists$ $M > 0$, $|y_n| \leq M$.
(and $M < \infty$). If $x = 0$, second term vanishes. This is a trivial case. Assume $x \neq 0$.
Choose $N_1 \Rightarrow n \geq N_1 \Rightarrow |x_n - x| < \dfrac{\varepsilon}{2M}$

$N_2 \Rightarrow n \geq N_2 \Rightarrow |y_n - y| < \dfrac{\varepsilon}{2|x|}$

Then if $N = \max\{N_1, N_2\}$

$$n \geq N \Rightarrow |x_n y_n - xy| \leq |y_n(x_n - x)| + |x(y_n - y)|$$

$$< \frac{|y_n| \varepsilon}{2M} + |x| \frac{\varepsilon}{2|x|}$$

$\dfrac{|y_n|}{M} < 1$ since $|y_n| \leq M$

$$< \varepsilon$$

(4) exercise

Theorem Every monotonically increasing (or decreasing) sequence, bounded above (or below) is convergent.

Proof If $x_{n+1} \geq x_n$ for all n, $\{x_n\}$ is monotone non decreasing. If $x_{n+1} > x_n$ all n, it is monotone increasing. (Similarly for decreasing)
Let $\{x_n\}$ be a bounded sequence with $x_{n+1} \geq x_n$, $A = \{x_1, x_2, \ldots\}$. A is non empty, bounded above. So it has a least upper bound. Call this x.
Choose $\varepsilon > 0$ and N such that

$x_N > x - \varepsilon$, because x is the least upper bound

Sequence is monotone. So $|x - x_N| < \varepsilon$, infact for all $n \geq N$

$$|x - x_n| = |x_n - x| < \varepsilon.$$

So $x_n \to x$.

The other case is similar

Example Let the sequence $\{x_n\}$ be given by $x_1 = 2.5$, $x_n = \frac{1}{5}(x_{n-1}^2 + 6)$ $n > 1$.

Show (i) $2 \leq x_n \leq 3$

(ii) $\{x_n\}$ is decreasing

(iii) $\{x_n\}$ is convergent

(1) Induction (Assume true etc)

(2)
$$x_{n+1} - x_n = \frac{1}{5}(x_n^2 + 6) - x_n$$
$$= \frac{1}{5}(x_n^2 - 5x_n + 6)$$
$$= \frac{1}{5}\underbrace{(x_n - 3)}_{\leq 0}\underbrace{(x_n - 2)}_{\geq 0}$$
$$\leq 0$$
$$x_{n+1} \leq x_n$$

Sequence decreases, bounded below so it converges.

Subsequences

Given a sequence $\{x_n\}$ we can define a subsequence $\{x_{n_k}\}$ where n_k is a sequence of natural numbers

eg. $\frac{1}{2n+1} = x_n$ $\quad n_k = k^2$

$$x_{n_k} = \frac{1}{2k^2+1}$$

$x_{n_k} = \{ \frac{1}{3}, \frac{1}{9}, \frac{1}{19}, \ldots \}$

<u>Theorem</u> Every sequence has a monotone subsequence.
<u>Proof</u> <u>Notes</u>

<u>Theorem</u> (Bolzano–Weierstrass)
Every bounded sequence has a convergent subsequence

<u>Proof</u> Every sequence has a monotone subsequence. (Proof by waffle)
Since it is bounded it is convergent

Every set with convergent subsequences is <u>compact</u>.

If a sequence converges, all subsequences converge to the same limit. If there are subsequences
x_{n_k}, x_{n_j} $x_{n_k} \to l$, $x_{n_j} \to m$
$l \neq m$
then sequence diverges
The limit superior and limit inferior
$\limsup_{n \to \infty} x_n$ = largest subsequence limit
$\liminf_{n \to \infty} x_n$ = smallest " limit
We have convergence if and only if
$\liminf x_n = \limsup x_n$
$x_n = (-1)^n$. $\limsup = 1$
$\liminf = -1$.

~~Theorem If a sequence is convergent all subsequences converge to the same limit.~~

~~Corollary if $\limsup x_n = \liminf x_n = l$~~
~~$x_n \to l$.~~

Theorem Suppose $x_n \to l$ an $n \to \infty$

(1) if $x_n \geq a$, $n = 1, 2, 3, \ldots$, then $l \geq a$

(2) if $x_n \leq b$, $n = 1, 2, 3, \ldots$, then $l \leq b$

Proof If $x_n \leq b$ then $-x_n \geq -b$ so we need only do (1)

Let $\varepsilon > 0$. Then there exists N such that for any $n > N$, $|x_n - l| < \varepsilon$

or $l - \varepsilon < x_n < l + \varepsilon$.

But $x_n \geq a$, so for any $n > N$

$a \leq x_n < l + \varepsilon$

Thus $a < l + \varepsilon \Rightarrow a \leq l$, which is (1).

Cauchy sequences A sequence $\{x_n\}$ is Cauchy if given any $\varepsilon > 0$, we can find N, such that

$n, m \geq N \Rightarrow |x_m - x_n| < \varepsilon$.

Every convergent sequence is Cauchy

Proof Pick N st. $|x_n - x| < \varepsilon/2$ if $n \geq N$.

$$\text{Then } |x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if $n, m \geq N$.