Geometric sums.

Let $S_N = a + ar + ar^2 + \dots + ar^N$

Then

$$rS_N = ar + ar^2 + \dots + ar^{N+1}$$

Hence $S_N - rS_N = (1-r)S_N = a - ar^{N+1}$.

Thus $S_N = \dfrac{a(1-r^{N+1})}{1-r}$.

Notice that if $|r|<1$, then $\lim_{N\to\infty} r^{N+1} = 0$.

So the infinite series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

provided that $|r|<1$; If $|r| \geq 1$, the series diverges. This has many applications. For example, if $r = -x^2$, $a=1$. Then

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \qquad |x|<1$$

This is an example of a Taylor Series.

$$\int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - x^2 + x^4 - \dots)\,dx$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Return to Cauchy sequences.

Recall a sequence is Cauchy if for every $\varepsilon > 0$, $\exists$ (there exists) $N \in \mathbb{N}$ such $n, m \geq N$ $|x_n - x_m| < \varepsilon$.

1.26

Prop If a Cauchy sequence has a convergent subsequence with limit x, then the Cauchy sequence converges to x

Proof Suppose $\{x_n\}$ is a Cauchy seq. Let $\{x_{n_k}\}$ be a subsequence, which converges to x.

[n_k is a sequence of natural numbers $n_k \to \infty$. $\{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$ is a subsequence]

Let $\varepsilon > 0$

There is an integer N such that

$n, m \geq N \Rightarrow |x_n - x_m| < \varepsilon/2$

Pick K such that $n_K > N$. Let $n > N$

$$|x_n - x| = |x_n - x_{n_K} + x_{n_K} - x|$$

$$\leq |x_n - x_{n_K}| + |x_{n_K} - x|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore x_n \to x$

Next

Exercise Every Cauchy sequence is bounded.

Theorem Every Cauchy sequence converges.

Proof Bolzano-Weierstrass says bounded $\Rightarrow$ convergent subsequence. Cauchy sequences are bounded, so they have convergent subsequences $\Rightarrow$ they converge by prop 1.26.

Let $S_N = \sum_{k=1}^{N} \frac{1}{k}$

Then $S_{N+1} - S_N = \frac{1}{N+1}$

$$|S_{N+1} - S_N| = \frac{1}{N+1} \to 0$$

$$\sum_{k=1}^{N} \frac{1}{N} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + -$$

$$\sum_{k=N}^{2N} \frac{1}{k} = \frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{2N}.$$

$$\geq N \times \frac{1}{2N} = \frac{1}{2}$$

Similarly $\sum_{k=2N+1}^{4N} \geq \frac{1}{2}$ $\sum_{4N+1}^{8N} \geq 1/2$

$\therefore \sum_{k=1}^{N} \frac{1}{k} \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

So sum diverges. $\therefore \{S_N\}$ is not Cauchy. $\sum_{n=1}^{\infty} \frac{1}{n}$ is the <u>harmonic</u> series. The number S_N come up a <u>lot</u>

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{n} - \ln n \right) = \gamma \text{ Euler's constant}$$

$\gamma \simeq .577\ldots$

<u>Example</u> Let $\{x_n\}$ be a sequence such that $|x_{n+1} - x_n| \leq \frac{1}{2^n}$.

Show this is Cauchy.

$$|x_{m+1} - x_n| = |x_{m+1} - x_m + x_m - x_n|$$
$$\leq |x_{m+1} - x_m| + |x_m - x_n|$$
$$\leq \frac{1}{2^m} + |x_m - x_n|$$

$$\le \frac{1}{2^m} + |x_m - x_{m-1} + x_{m-1} - x_n|$$

$$\le \frac{1}{2^m} + |x_m - x_{m-1}| + |x_{m-1} - x_n|$$

$$\le \frac{1}{2^m} + \frac{1}{2^{m-1}} + |x_{m-1} - x_n|$$

$$\le \frac{1}{2^m} + \frac{1}{2^{m-1}} + \cdot \qquad + \frac{1}{2^n}$$

$$= \frac{1}{2^n}\left(1 + \cdots + \frac{1}{2^{m-n}}\right)$$

$$= \frac{1}{2^n}\left(\frac{1-\left(\frac{1}{2}\right)^{m-n+1}}{1-1/2}\right) \to 0 \text{ as } m, n \to \infty.$$

<u>Infinite Series</u>

Let $S_N = \sum_{k=1}^{N} a_k$, a_k are real numbers

This is a sequence. We say that the series $\sum_{k=1}^{\infty} a_k$ converges if the sequence $\{S_N\}_{N=1}^{\infty}$ converges.

If $\sum_{k=1}^{\infty} |a_k| < \infty$. This is an <u>absolutely</u> convergent series.

If $\sum_{k=1}^{\infty} a_k < \infty$, but $\sum_{k=1}^{\infty} |a_k| = \infty$

the series is <u>conditionally</u> convergent.

Care is needed with cond. conv. series.

Famous examples

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \quad \text{(Euler)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

These are absolutely convergent

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \quad (*)$$

$$= \ln 2$$

But $\left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

So $(*)$ is <u>conditionally</u> convergent

Note $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} -$

$$\neq \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots\right)$$

<u>Theorem</u> If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are convergent with sums S and T then

$$\sum_{n=1}^{\infty} (a_n + b_n) = S + T$$

<u>Proof</u> (Notes)

<u>Note</u> $\sum_{n=1}^{\infty} \lambda a_n = \lambda S \quad (|\lambda| < \infty)$

<u>Lemma</u> If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. Proof notes

$$\left[\sum_{n=1}^{\infty} n = \infty, \quad n \to \infty\right]$$

<u>Proposition</u> If $\sum_{n=1}^{\infty} a_n$ is convergent then as $N, M \to \infty$ $\sum_{n=M+1}^{N} a_n \to 0$

<u>Proof</u> The sequence of partial sums $S_N = \sum_{n=1}^{N} a_n$ is convergent. So S_N is Cauchy. $|S_N - S_M| \to 0$ as $N, M \to \infty$

$$\text{i.e. } |S_N - S_M| = \left|\sum_{n=M+1}^{N} a_n\right| \to 0.$$

<u>Proposition</u> 1.33 If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

<u>Note</u> $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ is conditionally convergent [<u>Proof</u> later] Let $a_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n}}$

$$a_n b_n = \frac{1}{n}$$

$$\text{So } \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Proof of prop 1.33. Series $\sum_{n=1}^{\infty} a_n$ is convergent $\therefore \lim_{n\to\infty} a_n = 0$ $\left[S_N = \sum_{k=1}^{N} a_k b_k\right]$

$\therefore$ The sequence $\{a_n\}$ is bounded. Suppose $|a_n| \le K$. Then if $N > M$

$$|S_N - S_M| = \left|\sum_{k=1}^{N} a_k b_k - \sum_{k=1}^{M} a_k b_k\right|$$

$$= \left| \sum_{k=M+1}^{N} a_k b_k \right| \leq \sum_{k=M+1}^{N} |a_k b_k| \quad (*)$$

$$\left[\left| \sum_{k=M+1}^{N} a_k b_k \right| = |a_{M+1} b_{M+1} + a_{M+2} b_{M+2} + \cdots a_N b_N| \right.$$

Triangle inequality $\left. \leq |a_{M+1} b_{M+1}| + |\cdots\cdots| \right]$

$$(*) \leq K \sum_{k=M+1}^{N} |b_k| \to 0 \text{ as } M, N \to \infty$$

So $\{S_N\}$. So $S_N = \sum_{k=1}^{N} a_k b_k$

is convergent.