Real Analysis Workshop 4

Comparison Test Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have positive terms. If there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} a_n$ is convergent. Conversely if $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

Proof Let $T = \sum_{k=1}^{\infty} b_k < \infty$. Since the a_n's are positive $S_n = \sum_{k=1}^{n} a_k$, $n > N$ is increasing.

$$S_n = a_1 + a_2 + \ldots + a_N + a_{N+1} + \ldots + a_n$$

$$\leq a_1 + a_2 + \ldots + a_{N-1} + b_N + \ldots + b_n$$

$$= b_1 + b_2 + \ldots + b_n + (a_1 - b_1) + (a_2 - b_2) + \ldots + (a_{N-1} - b_{N-1})$$

Let $T_n = \sum_{k=1}^{n} b_k$. $T_n \to T$.

So $$S_n = T_n + \sum_{k=1}^{N-1} (a_k - b_k)$$
$$\leq \underbrace{T + \sum_{k=1}^{N-1} (a_k - b_k)}_{\text{upper bound}}.$$

$\therefore \{S_n\}$ converges so $\sum_{n=1}^{\infty} a_n$ converges

Proof of other case is an exercise.

<u>Example</u> $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. We know $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. We know $\sqrt{n} \le n$ all $n \ge 1$. So $\frac{1}{\sqrt{n}} \ge \frac{1}{n}$ all $n \ge 1$.

Hence $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges.

In fact $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ $0 \le \alpha \le 1$ diverges and we know $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

$n^3 \ge n^2$ for all $n \ge 1$.

Thus $\frac{1}{n^3} \le \frac{1}{n^2}$ all $n \ge 1$.

$\therefore \sum_{n=1}^{\infty}\frac{1}{n^3}$ converges by the comparison test.

In fact $\sum_{n=1}^{\infty}\frac{1}{n^p}$ $p > 1$ converges (later)

<u>The limit comparison test</u> Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of strictly positive terms. Suppose that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = l \ne 0$$

Then either both series converge or both diverge.

<u>Proof</u> <u>Notes.</u> Let us show how this works.

<u>Example</u> $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent

$$S_n = \sum_{k=1}^{n}\frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2}$$

$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$

$< 1 + \frac{1}{2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \dots \frac{1}{n(n-1)}$

$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \left(\frac{1}{n-1} - \frac{1}{n}\right)$

$= 2 - \frac{1}{n}. < 2.$

$\therefore$ S_n is increasing and bounded above.

Consider $\sum_{n=1}^{\infty} \underbrace{\frac{n+1}{2n^3+n+3}}_{a_n}$. $b_n = 1/n^2$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n+1}{2n^3+n+3} \Big/ \frac{1}{n^2}$

$= \frac{n^3+n^2}{2n^3+n+3} = \frac{n^3(1+\frac{1}{n})}{n^3(2+\frac{1}{n^2}+\frac{3}{n^3})}$

$\to \frac{1}{2} \neq 0.$

So both converge.

<u>The ratio test</u> Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive terms

Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Then the series converges if $L<1$. Diverges if $L>1$. If $L=1$, it is inconclusive.

<u>Proof</u> Let $a_n = \frac{1}{n}$. $\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \Big/ \frac{1}{n}$

$= \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1} \to 1.$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$a_n = \frac{1}{n^2}$, $\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)^2} \Big/ \frac{1}{n^2} = \frac{n^2}{(n+1)^2} = \left(1 - \frac{1}{n+1}\right)^2$

$\to 1$, but series converges

Suppose $L<1$. Pick $r \in \mathbb{R}$ such that $L<r<1$.
We can choose $N \in \mathbb{N}$ such that $n \geq N$ implies

$$\left|\frac{a_{n+1}}{a_n} - L\right| < r-L$$

or

$$-(r-L) < \frac{a_{n+1}}{a_n} - L < r-L. \quad (\ddagger)$$

So $\frac{a_{n+1}}{a_n} < r \Rightarrow a_{n+1} < r a_n$

Similarly $a_n < r a_{n-1} < r^2 a_{n-2} < r^3 a_{n-3}$ etc

So $a_n < r^k a_{n-k}$. Let $k = n-N$

Then $a_n \leq r^{n-N} a_N = \left(\frac{a_N}{r^N}\right) r^n$.

The series $\sum_{n=1}^{\infty} \left(\frac{a_N}{r^N}\right) r^n$ (*) is a geometric series.
The common ratio is r. $r<1$.
$\therefore$ The series (*) converges. By comparison test $\sum_{n=1}^{\infty} a_n$ converges.

For divergence take $L>1$, $1<r<L$
use $(\ddagger)$ again.

<u>nth root test</u> Let $\sum_{n=1}^{\infty} a_n$ be a series.

Suppose that $\limsup |a_n|^{1/n} = L$
If $L<1$ the series converges. If $L>1$ it diverges. If $L=1$ it is inconclusive.

<u>Proof Notes</u>

Examples. Ratio and nth root test

a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is convergent. Let $a_n = \frac{n}{e^n}$.

Then $\frac{a_{n+1}}{a_n} = \frac{n+1}{e^{n+1}} \Big/ \frac{n}{e^n}$

$[e^{a+b} = e^a e^b]$ $\quad = \frac{e^n}{e^{n+1}} \frac{n+1}{n} = \frac{1}{e}\left(1+\frac{1}{n}\right)$

Now $\lim_{n\to\infty} \frac{1}{e}\left(1+\frac{1}{n}\right) = \frac{1}{e} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)$

$= \frac{1}{e} < 1.$

So by the ratio test the series converges.

b) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$. This converges. We use nth root test.

$a_n = \left(\frac{n}{2n+1}\right)^n$

So $(a_n)^{1/n} = \left(\left(\frac{n}{2n+1}\right)^n\right)^{1/n} = \frac{n}{2n+1}$

$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$. So by nth root test, the series converges.

c) $\sum_{n=1}^{\infty} \frac{e^n}{n}$ diverges. $\frac{a_{n+1}}{a_n} = e \frac{n}{n+1} \to e > 1$.

d) $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$ $\quad a_n = \frac{2^n}{(2n)!}$. So $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(2(n+1))!} \Big/ \frac{2^n}{(2n)!}$

$= 2 \frac{(2n)!}{(2(n+1))!} = 2 \frac{(2n)!}{(2n+2)(2n+1)(2n)!}$

$= \frac{2}{(2n+2)(2n+1)} \to 0$ as $n \to \infty$

< 1, so series converges.

$(2(n+1))! = (2n+2)! = (2n+2)(2n+1)(2n)!$

<u>Alternating Series Test</u> Let $\{a_n\}$ be <u>a sequence</u> of positive terms $a_{n+1} \le a_n$ and $a_n \to 0$ as $n \to \infty$. Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

<u>Example</u> $a_n = \frac{1}{n} \to 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$
and $a_{n+1} < a_n$
is convergent.
$a_n = \frac{1}{\sqrt{n}} \to 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$
is convergent. Clearly $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$.

<u>Proof</u> We show that $S_N = \sum_{n=1}^{N} (-1)^{n+1} a_n$ is Cauchy. Let $\varepsilon > 0$. Since $a_n \to 0$ as $n \to \infty$, we can find $N \in \mathbb{N}$ such that $|a_n - 0| = |a_n| = a_n < \varepsilon$.
Since the $a_n \to 0$ monotically,

$$a_n - a_{n+1} \ge 0.$$

$$a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \dots + (-1)^{n+1} a_n \le a_{m+1}$$

If

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k \quad , \text{ pick } n > m \ge N$$

$$|S_n - S_m| = |a_1 - a_2 + a_3 - \dots + (-1)^{n+1} a_n - (a_1 - a_2 + \dots + (-1)^{m+1} a_m|$$

$$= |a_{m+1} - a_{m+2} + \quad (-1)^{n+1} a_n|$$

$$\le |a_{m+1}| < \varepsilon.$$

So $\{S_n\}$ is Cauchy, hence it converges.

<u>Cauchy Condensation Test</u> Suppose that the sequence a_n is positive and non increasing. Then the series $\sum_{n=1} a_n$ converges if and only if the series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$. $a_n = \frac{1}{n^p}$

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} \frac{1}{2^{np-n}}.$$

If $p>1$, this converges (It is a geometric sum). If $p \le 1$ it diverges. So $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p>1$, diverges for $p<1$.

Why does $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$?

Later we will see that for all $x \in \mathbb{R}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + -$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{(A Taylor series)}$$

Euler showed that

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \quad \text{(An infinite product)}$$

$$= x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) -$$

Now

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{5!} - -$$

$$= 1 - x^2\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + -\right)$$

$$= 1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)\frac{x^2}{\pi^2} -$$

$$= 1 - \frac{x^2}{6} + - - \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

<u>Example</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$

$$= 1.$$

More properly

Let $\displaystyle S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \cdots + \frac{1}{N} - \frac{1}{N} - \frac{1}{N+1}$

$$= 1 - \frac{1}{N+1} \to 1 \text{ as } N \to \infty.$$

So the sequence $\{S_N\}_{N=1}^{\infty}$ converges to 1 and hence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

This is an example of a <u>telescoping</u> series.

Let us consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$.

If $\alpha = 1$, the series diverges.

If ~~α~~ $\alpha < 1$, then $n^{\alpha} < n \Rightarrow \frac{1}{n} < \frac{1}{n^{\alpha}}$

So $\displaystyle\sum_{n=1}^{N} \frac{1}{n} < \sum_{n=1}^{N} \frac{1}{n^{\alpha}}$ and since the sequence on the left diverges, the sequence on the right does too.

What about the case $\alpha > 1$. This is an example of a convergent series. The key is to show that the sequence of partial sums is increasing and bounded above.

[Note the importance of our axiom here].

Clearly $\displaystyle\sum_{n=1}^{N+1} \frac{1}{n^\alpha} = \sum_{n=1}^{N} \frac{1}{n^\alpha} + \frac{1}{(N+1)^\alpha} > \sum_{n=1}^{N} \frac{1}{n^\alpha}$

So the sequence of partial sums is increasing. Now we have to show that it is bounded above.

Clearly

$$S_N = \sum_{n=1}^{N} \frac{1}{n^\alpha} \leq \sum_{n=1}^{2^N-1} \frac{1}{n^\alpha}$$

Similar idea to divergence of $\sum_{n=1}^{\infty} \frac{1}{n}$

$$= 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{(2^N-1)^\alpha}$$

$$= 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) + \left(\frac{1}{4^\alpha} + \frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha}\right) + \cdots + \left(\frac{1}{2^{(N-1)\alpha}} + \cdots + \frac{1}{(2^N-1)^\alpha}\right)$$

$$\leq 1 + \left(\frac{1}{2^\alpha} + \frac{1}{2^\alpha}\right) + \left(\frac{1}{4^\alpha} + \cdots + \frac{1}{4^\alpha}\right) + \cdots + \left(\frac{1}{2^{(N-1)\alpha}} + \cdots + \frac{1}{2^{(N-1)\alpha}}\right)$$

$$= 1 + \frac{2}{2^\alpha} + \frac{4}{4^\alpha} + \cdots + \frac{2^{N-1}}{2^{(N-1)\alpha}}$$

$$= 1 + \frac{1}{2^{\alpha-1}} + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \cdots + \left(\frac{1}{2^{\alpha-1}}\right)^{N-1}$$

$$= \frac{1 - (1/2^{\alpha-1})^N}{1 - (1/2^{\alpha-1})} \leq \frac{1}{1 - (1/2^{\alpha-1})}, \text{ since } \alpha > 1.$$

So the sequence is bounded above. Hence it converges. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty$ which is an important fact.