Real Analysis. Workshop 5

Functions, Limits and continuity.

A function $f: X \subseteq \mathbb{R} \to Y \subseteq \mathbb{R}$ is an operation or mapping that takes $x \in X$ to a unique $y \in Y$, $f(x) = y$.
X is called the domain

Y is called the range

The set $f(X) = \{ f(x) : x \in X \}$ is called the image of X under f.

Ex

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

This is the Dirichlet function

This is not continuous.

$$f(x) = x^2, \quad x \in \mathbb{R}.$$

This is continuous.

It turns out there are different kinds of continuity.

Limits: We want to define $\lim\limits_{x \to x_0} f(x)$

Definition: We define limit points and limits of functions

(1) A point x is a limit point of a set $X \subseteq \mathbb{R}$, if there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $x_n \to x$. If not x is an isolated point.

(2) Let $X \subseteq \mathbb{R}$, $f: X \to \mathbb{R}$ be a function and x_0 is a limit point of X. Then L is the limit of f as $x \to x_0$, if and only if given $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Ex

$X = [0,1] \cup [2,3]$ $\frac{1}{2}$ is a limit point of X

$X = [0,1] \cup \{2\}$ $\frac{1}{2}$ is a limit point

2 is an isolated point

Definition Let $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}$, we say $\lim_{x \to a+} f(x) = L$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x < a + \delta$, $|f(x) - L| < \varepsilon$. Similarly for $\lim_{x \to a^-} f(x)$

One can show that

$$\lim_{x \to a} f(x) = L$$

if and only if $\lim_{x \to a+} f(x) = L \lim_{x \to a^-} f(x)$

Proof (Exercise)

Defn Let $f : \mathbb{R} \to \mathbb{R}$, Then $\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon > 0$ we can find $M > 0$, such that $x \geq M \Rightarrow |f(x) - L| < \varepsilon$.

We say $\lim_{x \to -\infty} f(x) = L$ if we can find $M < 0$, such that $x < M \Rightarrow |f(x) - L| < \varepsilon$

Definition of continuity

Defn 1.(1) A function $f : X \to \mathbb{R}$ is said to be continuous at x if for any sequence $\{x_n\} \subseteq X$, $x_n \to x$ $\lim_{n \to \infty} f(x_n) = f(x)$

2) f is continuous at x, if given any $\varepsilon > 0$ here is a $\delta_x > 0$ such that $|x - y| < \delta_x \Rightarrow |f(x) - f(y)| < \varepsilon$

<u>Theorem</u> These two definitions are equivalent, ie. They imply each other

<u>Proof</u> <u>Notes</u>

An open set is $(0,1) = A$

$x = 1/2 \in A$, $x \in (\frac{1}{4}, 3/4) \subset A$.

$B = [0,1]$ closed. $0 \in B$. Any open interval containing 0, will have some part outside B, so B is not open

<u>Definition</u> A function $f: X \to Y$ is continuous if for every open set $B \in Y$, the inverse image

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

is itself an open set.

<u>Ex</u> $f = x^3$ $\quad f^{-1}((0,1)) = (0,1)$ which is open.

<u>Theorem</u> (1) If f, g are continuous, a, b are constants af, $af + bg$, fg and $\frac{f}{g}$ (provided g is non zero are continuous)

(2) Suppose that $g: X \subseteq \mathbb{R} \to Y \subseteq \mathbb{R}$ and $f: Y \to Z \subseteq \mathbb{R}$ are continuous. Then $f \circ g: X \to Z$ is also continuous

$(f \circ g)(x) = f(g(x))$

<u>Proof</u> $(f \circ g)^{-1}(B)$

$= g^{-1}(f^{-1}(B))$

B is open, so $f^{-1}(B) = A$ is open.

$g^{-1}(f^{-1}(B))$

$= g^{-1}(A)$

$=$ which is open

$\therefore (f \circ g)^{-1}(B)$ is open $\Rightarrow f \circ g$ is continuous

[A B matrices

$(AB)^{-1} = B^{-1}A^{-1}$

$B^{-1}A^{-1}(AB)$

$= B^{-1}(A^{-1}A)B$

$= B^{-1}IB$

$= B^{-1}B = I$]

A function can be continuous to the left or right
eg. f is right continuous at $x=a$
if $\lim_{x \to a^+} f(x) = f(a)$.

<u>Uniform Continuity</u> There are 3 crucial properties of continuous functions.

(1) Every continuous function $f:[a,b] \to \mathbb{R}$ is uniformly continuous
(2) Every continuous function on $[a,b]$ attains its maximum and minimum values on $[a,b]$
(3) f has the intermediate value property.

<u>Definition</u> A function $f: X \to \mathbb{R}$ is said to be uniformly continuous if given $\varepsilon > 0$, we can find $\delta > 0$, such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Suppose f is uniformly continuous and ~~Sosid~~ $|x-y| < \frac{1}{99}$

$$\Rightarrow |f(x) - f(y)| < \varepsilon$$

Now $\left|\frac{101}{100} - 1\right| = \frac{1}{100} < \frac{1}{99}$.

$$\therefore \left|f\left(\frac{101}{100}\right) - f(1)\right| < \varepsilon$$

Also $\left|100\frac{1}{100} - 100\right| = \frac{1}{100} < \frac{1}{99}$

$$\Rightarrow \left|f\left(100\tfrac{1}{100}\right) - f(100)\right| < \varepsilon$$

The point is $|f(x) - f(y)| < \varepsilon$ whenever $|x-y| < \frac{1}{99}$. It does not matter where x and y are on $\mathbb{R}$.

(<u>Note</u> A closed and bounded interval has the form $[a,b]$, a,b finite)

To prove (I) we use an equivalent definition

Defn A function $f: X \to \mathbb{R}$ is sequentially uniformly continuous if given $x_n, y_n \in X$, $y_n - x_n \to 0$ $\Rightarrow f(y_n) - f(x_n) \to 0$

This is equivalent to uniform continuity.

Theorem A continuous function on a closed and bounded interval is uniformly continuous.

Proof Suppose that is not uniformly continuous. It therefore cannot be seq uniformly continuous. Choose $r > 0$ such that for every $\delta > 0$ there exists $x, y \in [a,b]$ such that

$$|x - y| < \delta \text{ and } |f(x) - f(y)| > r$$

For each $n \in \mathbb{N}$ choose $x_n, y_n \in [a,b]$ such that

$$|x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| > r.$$

x_n, y_n are bounded. By BW $\{x_n\}$ and $\{y_n\}$ have convergent subseq. x_{n_k}, y_{n_k}. $x_{n_k} \to x$ as $n_k \to \infty$

$y_{n_k} \to y$ as $n_k \to \infty$

$\{x_{n_k} - y_{n_k}\}$ is a conv. subseq of $\{x_n - y_n\}$. Now $x_n - y_n \to 0$

So $x_{n_k} - y_{n_k} \to 0$

$\therefore y_{n_k} = x_{n_k} - (x_{n_k} - y_{n_k}) \to x = y$

f is continuous $\therefore f(x_{n_k}) \to f(x)$

$f(y_{n_k}) \to f(x)$

We assumed $|f(x) - f(y)| > r$

But $|f(x_{n_k}) - f(y_{n_k})| \to 0$

This is a contradiction. It would mean f is not continuous

Example $\sin x$ is continuous on $\mathbb{R}$

$$|\sin x - \sin y| = \left|2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right|$$

$$\leq \left|2\sin\left(\frac{x-y}{2}\right)\right|$$

$$(|\sin z| \leq |z|) \qquad \leq 2\left|\frac{x-y}{2}\right|$$

$$= |x-y|$$

$\delta = \varepsilon$ Then $|x-y| < \delta$

$\Rightarrow |\sin x - \sin y| < \varepsilon$

Maxima and Minima The maximum value of a function : $f: X \to \mathbb{R}$ is the smallest number M such that $f(x) \leq M$ for all $x \in X$. A minimum value is the largest m such that $f(x) \geq m$ all $x \in X$.

Theorem A continuous function on a closed and bounded interval $[a,b]$ attains its maximum and minimum values on $[a,b]$.

i.e. f is bounded on $[a,b]$ and it attains its upper and lower bounds.

Example $f(x) = x^2$ on $[0,1]$

upper bound $=1$, $f(1)=1$, lower bound is 0, $f(0)=0$.

$f(x) = x^2$, $[0,1)$. Upper bound $= 1$, but $f(x) \neq 1$ for any $x \in [0,1)$.

So result is false if the interval is not closed and bounded.