A useful fact about Riemann sums is that if f is Riemann integrable on $[a,b]$, $a = x_0 < x_1 < \ldots < x_n = b$ is a partition of $[a,b]$ and $c_i \in [x_{i-1}, x_i]$ then

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \to \int_a^b f(x)\,dx \quad \text{as}$$

$\max |x_i - x_{i-1}| \to 0$

However we will not prove this.

Double integrals. We make only brief remarks.

If $f : [a,b] \times [c,d] \to \mathbb{R}$

we define the double integral by

$$\int_R f\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

where $R = [a,b] \times [c,d]$

If $f \geq 0$ we can interpret the integral as the volume beneath the surface f over the rectangle R.

The double integral exists if f is continuous. The most important fact is

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

This is called Fubini's Theorem.

We can also define the integral over an arbitrary continuous region, but we do not need this for our purposes

Example $f(x,y) = x^2 + y^2$. $[a,b] = [0,1]$, $[c,d] = [0,1]$

$$\int_0^1 \int_0^1 (x^2 + y^2)\,dx\,dy = \int_0^1 \left[\frac{x^3}{3} + x y^2 \right]_0^1 dy = \int_0^1 \left(\frac{1}{3} + y^2 \right) dy$$

$$= \left[\frac{1}{3} y + \frac{y^3}{3} \right]_0^1 = 2/3.$$

Pointwise and Uniform Convergence

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions. We say that $f_n \to f$ on X, (converges to f) pointwise, if for every $x \in X$,

$$f(x_n) \to f(x).$$

That is, given $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that if $n \geq N$

$$|f_n(x) - f(x)| < \varepsilon.$$ Note N may depend on x.

Let f be an analytic function on I and

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}, \quad a \in I$$

Then $$f_N(x) = \sum_{n=0}^{N} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

$f_N \to f$ pointwise on I.

Example $f_n(x) = \dfrac{n^2 x^2}{n^2+1}$, $f(x) = x^2$

$$f_n(x) = \frac{(n^2+1-1)x^2}{n^2+1} = x^2 - \frac{x^2}{n^2+1}$$

$$|f_n(x) - f(x)| = \left| x^2 - \frac{x^2}{n^2+1} - x^2 \right|$$

$$= |x|^2 \frac{1}{n^2+1} \to 0 \quad \text{all } x$$

So $f_n \to f$ pointwise

$$f_{n,j} = (\cos(n!\,\pi x))^{2j}, \qquad x \in [0,1]$$

$$\to \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \notin \mathbb{Q} \end{cases} \quad (p60)$$

If $f_n \to f$ is it true that
$\int f_n \to \int f$?

In general <u>No</u>

$$\int_0^1 f_{n,j} \not\to \int_0^1 D(x)\,dx \quad \text{where}$$

$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. In fact the Riemann integral of $D(x)$ does <u>not</u> <u>exist</u>

<u>Example</u> $f_n(x) = nxe^{-nx^2}$ on $[0,1]$

$$\int_0^1 nxe^{-nx^2}\,dx = \frac{1-e^{-n}}{2} \to \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{nx}{e^{nx^2}} = 0 \quad \text{for all } x \in [0,1]$$

So $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$

Thus $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$

When does $\lim_{n\to\infty} \int f_n = \int \lim f_n$?

If $f_n \to f$ <u>uniformly</u>

<u>Definition</u> A sequence of functions $\{f_n\}$ on a set $X \subseteq \mathbb{R}$ converges uniformly to f on X if for any $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $n \geq N$ implies
$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in X$$

$f_n(x) = \dfrac{n^2 x^2}{n^2+1} \qquad x \in [-1,1]$

Let $\varepsilon > 0 \qquad f(x) = x^2$

$$|f_n(x) - f(x)| = |x|^2 \frac{1}{n^2+1} \leq \frac{1}{n^2+1}.$$

$n^2+1 > n$

$$\therefore \frac{1}{n^2+1} < \frac{1}{n}.$$

If $N > 1/\varepsilon$ then for all $x \in [-1,1]$

$$|f_n(x) - f(x)| \leq \frac{1}{n^2+1} < \frac{1}{n} < \varepsilon$$

So $f_n \to f$ uniformly

Example $f_n(x) = \sqrt{x^2 + 1/n^2} \to |x| = f(x)$

Let $\varepsilon > 0$

$$\sqrt{x^2+\frac{1}{n}} - |x| = \left(\sqrt{x^2+\frac{1}{n^2}} - |x|\right) \frac{\sqrt{x^2+\frac{1}{n^2}} + |x|}{\sqrt{x^2+\frac{1}{n^2}} + |x|}$$

$$= \frac{x^2+\frac{1}{n^2} - x^2}{\sqrt{x^2+\frac{1}{n^2}} + |x|} = \frac{1}{n^2\left(\sqrt{x^2+\frac{1}{n^2}} + |x|\right)}$$

$$\leq \frac{1}{n}$$ (take $x=0$ to make denominator as small as possible)

If $N > \frac{1}{\varepsilon}$, $n \geq N$

$$\Rightarrow |f_n(x) - f(x)| < \frac{1}{n} < \varepsilon$$

for all x. So $f_n \to f$ uniformly

Lemma If $f_n \to f$ uniformly on $X \subseteq \mathbb{R}$, $f_n \to f$ pointwise

proof This is trivial

Theorem If $\{f_n\}_{n=1}^{\infty}$ is a uniformly convergent sequences of continuous functions on $X \subseteq \mathbb{R}$, with $f_n \to f$ then, f is continuous on X

Proof Since f_k is continuous at $x \in X$ given $\varepsilon > 0$, we may choose $\delta > 0$ such that $0 < |x-y| < \delta$ we have

$$|f_k(x) - f_k(y)| < \varepsilon/3$$

By uniform convergence, we may choose $N \in \mathbb{N}$ such that $k \geq N$ implies

$$|f(x) - f_k(x)| < \varepsilon/3$$

for all $x \in X$. Given $x \in X$, then for all $y \in X$, $0 < |y-x| < \delta$ we have

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| \\ &\quad + |f_k(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

$\therefore f$ is continuous

Proposition If $f_n \to f$ uniformly on X and $g_n \to g$ uniformly on X, then

$$af_n + bg_n \to af + bg$$

uniformly

<u>Proposition</u> If $f_n \to f$, uniformly on $[a,b]$
$g_n \to g$ uniformly, on $[a,b]$
$g_n f_n \to fg$ uniformly
on $[a,b]$, provided interval is bounded.

This is <u>not</u> true on the whole of $\mathbb{R}$.

<u>The Weierstrass M test</u> Let $\{f_n\}$ be a sequence of functions on $X \subseteq \mathbb{R}$
Such that $|f_n(x)| \leq M_n$ all $x \in X$
and $\sum_{n=1}^{\infty} M_n < \infty$

Then the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on X

<u>Proof</u> Let

$$S_N(x) = \sum_{n=1}^{N} f_n(x),$$

Suppose $|f_n(x)| \leq M_n$. Then if $N > K$

$$|S_N(x) - S_K(x)| = \left|\sum_{n=K+1}^{N} f_n(x)\right|$$
$$\leq \sum_{n=K+1}^{N} |f_n(x)|$$
$$\leq \sum_{n=K+1}^{N} M_n \to 0$$

as $K, N \to \infty$, since $\sum_{n=1}^{\infty} M_n < \infty$

$\therefore$ $\{S_N\}$ is a Cauchy sequence for all x. So it converges pointwise
N, K don't depend on x, so convergence is uniform.

<u>Example</u> $f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+1}$.

Here $f_n(x) = \frac{\cos(nx)}{n^2+1}$. f_n is continuous

$$|f_n(x)| = \frac{|\cos(nx)|}{n^2+1} \le \frac{1}{n^2+1}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \infty$.

So $\sum_{n=1}^{\infty} f_n$ is uniformly convergent

So f is continuous.

Swapping limits and Integrals

<u>Theorem</u> If $\{f_n\}_{n=1}^{\infty}$ is a sequence of Riemann integrable functions converging uniformly to f on $[a,b]$, then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$
$$= \int_a^b f(x)\,dx.$$

<u>Proof</u> If f is integrable, this is easy

Let $\varepsilon > 0$. Choose N, s.t. $n \ge N$

$$\Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \text{all } x \in [a,b)$$

Then for $n \ge N$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx$$
$$< \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon$$

Now we show that f is integrable.

Each f_n is bounded, so the limit f is bounded. Pick $\varepsilon > 0$, by uniform convergence we can choose $N \in \mathbb{N}$, $n \geq N$

$$\Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}.$$

Since f_N is integrable, Riemanns criterion says that we can find a partition P of $[a,b]$ with

$$U(f_N, P) - L(f_N, P) < \frac{\varepsilon}{3}.$$

Now

$$\sup_{x \in [a,b]} |f_N(x) - f(x)| < \frac{\varepsilon}{3(b-a)}$$

$$\begin{aligned} U(f, P) - L(f, P) &= U(f + f_N - f_N, P) \\ &\quad - L(f + f_N - f_N, P) \\ &= U(f_N, P) + U(f - f_N) \\ &\quad - L(f_N, P) - L(f - f_N, P) \\ &= U(f_N, P) - L(f_N, P) \\ &\quad + U(f - f_N, P) \\ &\quad - L(f - f_N, P) \\ &< \frac{\varepsilon}{3} + \frac{2(b-a)}{3(b-a)} = \varepsilon \end{aligned}$$

So f is Riemann integrable.

To clarify

$$U(f, P) = \sum_{i=1}^{n} f(\bar{x}_i)(x_i - x_{i-1})$$

where $f(\bar{x}_i)$ is the largest value of f on $[x_{i-1}, x_i]$

$$U(f+f_N-f_N) = \sum_{i=1}^{n} \left(f(\bar{x}_i) + f_N(\bar{x}_i) - f_N(\bar{x}_i)\right)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} f_N(x_i)(x_i - x_{i-1})$$

$$+ \sum_{i=1}^{N} (f(x_i) - f_N(x_i))(x_{i-1})$$

$$= U(f_N, P) + U(f-f_N, P)$$

Now

$$U(f-f_N, P) \leq \sum_{n=1}^{n} |f-f_N|(x_i - x_{i-1})$$

$$< \frac{\varepsilon}{3(b-a)} \sum_{i=1}^{n} (x_i - x_{i-1})$$

$$= \frac{\varepsilon}{3(b-a)}(b-a)$$

Since

$$\sum_{i=1}^{n} x_i - x_{i-1} = x_1 - a + x_2 - x_1 + \cdots + b$$

$$= b-a$$

Similar comments apply for $L(f, P)$

Note $U(f-f_N, P) \geq L(f-f_N, P)$ by definition.

Clearly $U(f-f_N, P) - L(f-f_N, P) \leq 2U(f-f_N, P)$

$$= \frac{2\varepsilon}{3}.$$