A sequence of functions $\{f_n\}$ on $X \subseteq \mathbb{R}$ is <u>uniformly Cauchy</u> on X if for any $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $m, n \geq N$

$$|f_n(x) - f_m(x)| < \varepsilon$$

for all $x \in X$.

<u>Theorem</u> A sequence of functions $\{f_n\}$ on $X \subseteq \mathbb{R}$ is uniformly convergent on X if and only if it is uniformly Cauchy on X.

<u>Proof</u> See notes.

Now we turn to the problem of swapping limits and derivatives

<u>Theorem</u> Let I be an open interval in $\mathbb{R}$. Let $\{f_n\}$ be a sequence of functions on I which converge uniformly to f on I. Let the sequence of derivatives f_n' converge uniformly to g on I. Then f is differentiable on I and $f'(x) = g(x)$ for all $x \in I$

<u>Proof</u> Let $\varepsilon > 0$ and pick an $N_1 \in \mathbb{N}$ such that

$$\sup_{x \in I} |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad (\text{for } n \geq N_1)$$

The sequence $\{f_n'\}$ is uniformly Cauchy on I.

So we can find $N_2 \in \mathbb{N}$ such that for all $x \in I$, $n, m \geq N_2$

$$|f_n'(x) - f_m'(x)| < \frac{\varepsilon}{3} \qquad (*)$$

Now let $N = \max\{N_1, N_2\}$.

The function f_N is differentiable on I and so at any point $x_0 \in I$ there exists $\delta > 0$ such that for $x \in I$, $0 < |x - x_0| < \delta$

$$\left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) \right| < \frac{\varepsilon}{3}$$

Now let $x \in I$, $x \neq x_0$, choose $M \geq N$. $f_M - f_N$ is differentiable and so by the Mean Value theorem we can find c between x and x_0 such that

$$\frac{(f_M - f_N)(x) - (f_M - f_N)(x_0)}{x - x_0} = (f_M - f_N)'(c)$$

or

$$(f_M - f_N)(x) - (f_M - f_N)(x_0) = (f_M - f_N)'(c)(x - x_0)$$

Thus

$$|f_M(x) - f_M(x_0) - (f_N(x) - f_N(x_0))| = |f_M'(c) - f_N'(c)|\,|x - x_0|$$
$$< \frac{\varepsilon}{3}|x - x_0|$$

by (*). Taking limits as $M \to \infty$ we have

$$|f(x) - f(x_0) - (f_N(x) - f_N(x_0))| \leq \frac{\varepsilon}{3}|x - x_0|.$$

Hence

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{(f_N(x) - f_N(x_0))}{x - x_0} \right| \leq \frac{\varepsilon}{3}.$$

Now we combine these. Let $x \in I$, $0 < |x - x_0| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| \leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{(f_N(x) - f_N(x_0))}{x - x_0} \right|$$
$$+ \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) \right|$$
$$+ |f_N'(x_0) - g(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

So f' exists and $f'(x_0) = g(x_0)$.

Note that we added and subtracted $\frac{f_N(x)-f_N(x_0)}{x-x_0}$ and $f_N'(x_0)$ and applied the triangle inequality

Before turning to our final topic, we mention a result that points to the future.

Theorem Assume that $\{f_n\}$ is a uniformly bounded, pointwise convergent sequence on $[a,b]$ and suppose that each f_n is Riemann-integrable on $[a,b]$. Suppose that $f = \lim_{n\to\infty} f_n$ is Riemann integrable on $[a,b]$. Then $\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$

(1885)

This is due to Arzela. The proof is quite hard and we omit it.

Note Uniformly bounded means that there is a constant $B>0$ such that for all $n>0$ and $x\in[a,b]$

$$|f_n(x)| \leq B$$

Methods of Integration

We start with the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad z>-1$$

Note $\Gamma(z+1) = z\Gamma(z)$. (Integrate by parts)

$\Gamma(1) = \int_0^\infty e^{-t}dt = 1$

$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = \Gamma(1)$, $\Gamma(3) = \Gamma(1+2) = 2\Gamma(1) = 2$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\times2\times1 = 3!$$

In general

$$\Gamma(n+1) = n!$$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{1/2-1}e^{-t}dt = \int_0^\infty t^{-1/2}e^{-t}dt$$

Put $t = u^2$, $dt = 2u\,du$ $\quad t^{-1/2} = \frac{1}{u}$

$$\therefore \Gamma(1/2) = 2\int_0^\infty u\, e^{-u^2}\frac{du}{u}$$

$$= 2\int_0^\infty e^{-u^2}du = \sqrt{\pi} \text{ — later.}$$

Some properties: $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$
$0 < z < 1$

2) $\Gamma(z-n) = (-1)^{n-1}\dfrac{\Gamma(-z)\Gamma(1+z)}{\Gamma(n+1-z)}$

3) $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$

4) $\displaystyle\prod_{k=0}^{m}\Gamma\left(z+\frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz}\,\Gamma(mz)$

5) $\dfrac{1}{\Gamma(z)} = ze^{\gamma z}\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) \simeq .577216$$

The Beta function is

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \qquad a,b>0$$

Theorem $B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Example $I = \int_0^1 t^{-1/2}(1-t)^{1/2}\,dt$

$$= B\left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

$$= \frac{\Gamma(1/2)\,\Gamma(3/2)}{\Gamma(1/2+3/2)}$$

$\Gamma(3/2) = \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma(1/2)$, $\Gamma(1/2) = \sqrt{\pi}$

$$= \frac{\sqrt{\pi}}{2}$$

$\Gamma(2) = 1$

$$\therefore\ I = \frac{\sqrt{\pi}\,\frac{\sqrt{\pi}}{2}}{1} = \frac{\pi}{2}$$

Example $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$

Put $u = \sin^2 x$. Then $\sqrt{\sin x} = u^{1/4}$.

$du = 2\sin x\cos x\,dx$. $\quad x = \frac{\pi}{2},\ u = 1$

$\cos x = \sqrt{1-\sin^2 x}$

$= (1-u)^{1/2}$ $\quad x = 0,\ u = 0$

$dx = \frac{1}{2}\frac{1}{\sin x\cos x}\,du$

So $I = \frac{1}{2}\int_0^1 u^{1/4}/u^{1/2}(1-u)^{1/2}\,du$

$$= \frac{1}{2}\int_0^1 u^{-\frac{1}{4}}(1-u)^{-1/2}\,du$$

$$= \frac{1}{2}B\left(\tfrac{3}{4}, \tfrac{1}{2}\right) = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{4})}.$$

Simple substitutions

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^4 x} = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\, dx$$

Put $x = \frac{\pi}{2} - u \qquad dx = -du$

$$\cos^4\left(\frac{\pi}{2} - u\right) = \sin^4 u$$
$$\sin^4\left(\frac{\pi}{2} - u\right) = \cos^4 u$$

$$\therefore \quad I = -\int_{\frac{\pi}{2}}^{0} \frac{\sin^4 u}{\cos^4 u + \sin^4 u}\, du$$
$$= \int_0^{\pi/2} \frac{\sin^4 x}{\cos^4 x + \sin^4 x}\, dx$$

$$\therefore \quad I + I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x}\, dx$$
$$= \int_0^{\pi/2} dx = \frac{\pi}{2}$$

$$\therefore \quad I = \frac{\pi}{4}$$

Example

$$I = \int_{-\frac{\pi}{2}}^{\pi/2} \frac{dx}{(1+e^{x\cos^2 x})(\cos^4 x + \sin^4 x)} = \frac{\pi}{\sqrt{2}}$$

$x = -u \qquad I + I = \int_{-\pi/2}^{\pi/2} \frac{dx}{\cos^4 x + \sin^4 x}$ Notes

Differentiation under the integral sign

Let $F(t) = \int_a^b f(x,t)\, dx$

Then $F'(t) = \int_a^b \frac{\partial f}{\partial t}(x,t)\, dx$

Example Evaluate $\int_0^1 \frac{x-1}{\ln x}\, dx$

Let $F(t)=\int_0^1 \frac{x^t-1}{\ln x}\,dx$

$x^t = e^{t\ln x}$ $\therefore$ $\frac{d}{dt}x^t = \ln x\, e^{t\ln x}$
$= x^t \ln x$

$\therefore$ $F'(t) = \int_0^1 \frac{\partial}{\partial t}\left(\frac{x^t-1}{\ln x}\right)dx$
$= \int_0^1 \frac{x^t\ln x}{\ln x}\,dx = \int_0^1 x^t\,dx$
$= \frac{x^{t+1}}{t+1}\Big|_0^1 = \frac{1}{t+1}$

$\therefore$ $F'(t)=\frac{1}{t+1}$ Thus $F(t)=\ln(t+1)+C$

Now $F(0)=\int_0^1\frac{x^0-1}{\ln x}\,dx = \int_0^1 \frac{1-1}{\ln x}\,dx = 0$

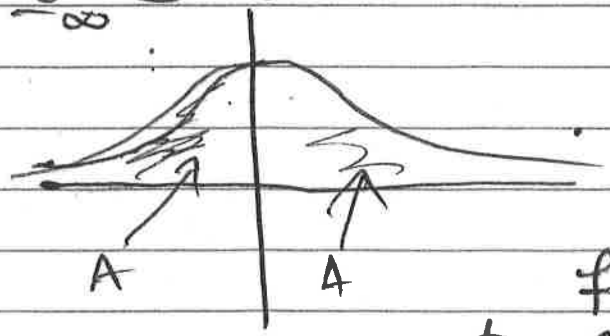
So $F(0)=\ln(1+0)+C=0$ $\therefore$ $C=0$

Hence $F'(t)=\ln(1+t)$

We want $\int_0^1 \frac{x-1}{\ln x}\,dx = F(1)=\ln 2$

Example Show $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$

$\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^{\infty} e^{-x^2}dx$



$f(t)=\left(\int_0^t e^{-x^2}dx\right)^2$

$f'(t)=2\left(\int_0^t e^{-x^2}dx\right)e^{-t^2}$
$=2\int_0^t e^{-(x^2+t^2)}dt$

Put $x = yt$, $f'(t) = 2t \int_0^1 e^{-(1+y^2)t^2} dy$

$$= -\frac{d}{dt} \int_0^1 \frac{e^{-(1+y^2)t^2}}{(1+y^2)} dy$$

Put $g(t) = \int_0^1 \frac{e^{-(1+y^2)t^2}}{1+y^2} dy$

Then $f'(t) = -g'(t)$ $\therefore f' + g' = 0$

Thus $f(t) + g(t) = \text{const}$

Now $f(0) + g(0) = 0 + \int_0^1 \frac{dy}{1+y^2}$

$$= \tan^{-1} y \Big|_0^1 = \frac{\pi}{4}$$

So $\left(\int_0^t e^{-x^2} dx \right)^2 + \int_0^1 \frac{e^{-(1+y^2)t^2}}{1+y^2} dy = \frac{\pi}{4}$

$$\lim_{t \to \infty} \int_0^1 \frac{e^{-(1+y^2)t^2}}{1+y^2} dy = \int_0^1 \lim_{t \to \infty} \frac{e^{-(1+y^2)t^2}}{1+y^2} dy$$

$$= \int_0^1 0 \, dy = 0$$

$$\lim_{t \to \infty} f(t) = \left(\int_0^\infty e^{-x^2} dx \right)^2$$

$$\therefore \lim_{t \to \infty} \left(f(t) + g(t) \right) = \lim_{t \to \infty} \frac{\pi}{4}$$

$$\therefore \left(\int_0^\infty e^{-x^2} dx \right)^2 = \frac{\pi}{4}$$

or $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Find $\int_0^\infty \frac{\sin x}{x}\,dx$

Note $\int_0^\infty e^{-xy}\,dy = \frac{1}{x}$

So $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \int_0^\infty \sin x\, e^{-xy}\,dy\,dx$

$$= \int_0^\infty \int_0^\infty \left(\sin x\, e^{-xy}\,dx\right) dy$$

$$= \int_0^\infty \frac{1}{1+y^2}\,dy \quad \text{(Tute question)}$$

$$= \tan^{-1} y \Big|_0^\infty = \frac{\pi}{2}$$

Find $\int_0^\infty e^{-xt} \frac{\sin x}{x}\,dx$

Let $F(t) = \int_0^\infty e^{-xt} \frac{\sin x}{x}\,dx$

$$F'(t) = \int_0^\infty -x e^{-xt} \frac{\sin x}{x}\,dx$$

$$= -\int_0^\infty e^{-xt} \sin x\,dx$$

$$= -\frac{1}{1+t^2}$$

$$F(t) = -\tan^{-1} t + C$$

$$F(0) = -\tan^{-1} 0 + C = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

$$\therefore \int_0^\infty e^{-xt} \frac{\sin x}{x}\,dx = \frac{\pi}{2} - \tan^{-1} t.$$