There are several ways that differentiation under the integral sign can be used.
Sometimes we need to solve a simple differential equation.

<u>Example</u> Calculate

$$F(t) = \int_0^\infty e^{-x^2}\cos(2xt)\,dx$$

First note that

$$F'(t) = -\int_0^\infty 2x e^{-x^2}\sin(2xt)\,dx$$

$$= \int_0^\infty \frac{d}{dx}\left(e^{-x^2}\right)\sin(2xt)\,dx$$

$$= \underbrace{\cancel{\left[e^{-x^2}\sin(2xt)\right]_0^\infty}}_{0} - 2t\int_0^\infty e^{-x^2}\cos(2xt)\,dx$$

$$= -2tF(t)$$

$$F'(t) + 2tF(t) = 0$$

So $\quad \dfrac{dF}{dt} = -2tF$

or $\quad \dfrac{dF}{F} = -2t\,dt \Rightarrow \displaystyle\int \frac{dF}{F} = -\int 2t\,dt$

Integrating gives

$$\ln F = -t^2 + C$$

or

$$F(t) = e^{-t^2}e^{c} = Ae^{-t^2} \qquad A = e^{c}$$

Now $F(0) = \displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} = A$

So $\quad F(t) = \dfrac{\sqrt{\pi}}{2}e^{-t^2}.$

$$I = \int_0^\infty e^{-xt}\left(\frac{\cos(ax)-\cos(bx)}{x}\right)dx$$

<u>Note</u> $\displaystyle\int_a^b \sin(xy)\,dy = -\frac{\cos(xy)}{x}\Big]_a^b$

$$= \frac{\cos(ax)-\cos(bx)}{x}$$

$$\therefore\ I = \int_0^\infty e^{-xt}\int_a^b \sin(xy)\,dy\,dx$$

$$= \int_a^b\left(\int_0^\infty e^{-xt}\sin(xy)\,dx\right)dy$$

$$= \int_a^b \frac{y}{y^2+t^2}\,dy$$

$$= \frac{1}{2}\ln(y^2+t^2)\Big|_a^b$$

$$= \frac{1}{2}\left[\ln(b^2+t^2)-\ln(a^2+t^2)\right]$$

$$= \frac{1}{2}\ln\left(\frac{b^2+t^2}{a^2+t^2}\right)$$

$$= \ln\left(\frac{b^2+t^2}{a^2+t^2}\right)^{\frac{1}{2}}$$

<u>Series Methods</u> An important method is to replace a function by a series

<u>Example.</u> Calculate $\displaystyle I = \int_0^1 \frac{\ln x}{x+1}\,dx$

Notice $\dfrac{1}{x+1} = 1-x+x^2-x^3+\ldots$

$|x|<1$

<u>idea:</u> integrate $\displaystyle\int_0^1 \ln x(1-x+x^2-x^3+\ldots)\,dx$

$\ln x$ is not bounded on $[0,1]$. No problem

Let $f_N(x) = \ln x \sum_{n=1}^{N} (-1)^n x^n$

$$\to \ln x\left(\frac{1}{1+x} - 1\right)$$

Each f_N is Riemann integrable on $[0,1]$ (Exercise). The sequence is uniformly bounded

So we have $\lim_{N\to\infty} \int_0^1 f_N \to \int_0^1 \ln x\left(\frac{1}{1+x} - 1\right) dx$

we have

$$\int_0^1 \frac{\ln x}{x+1}\,dx = \int_0^1 \ln x\,dx + \lim_{N\to\infty}\int_0^1 f_N$$

$$\int_0^1 \ln x\,dx = x\ln x\Big|_0^1 - \int_0^1 x\cdot\frac{1}{x}\,dx$$

$$= -1$$

$\left(\lim_{x\to 0} x\ln x = 0\right)$

(Use L'Hôp $\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}$ has form $\frac{\infty}{\infty}$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to 0} \frac{-x^2}{x} = -\lim_{x\to 0} x = 0 \Big)$$

$$\int_0^1 x^n \ln x\,dx = \frac{x^{n+1}\ln x}{n+1}\Big]_0^1 - \int_0^1 \frac{x^{n+1}}{(n+1)}\frac{1}{x}\,dx$$

$$= -\int_0^1 \frac{x^n}{(n+1)}\,dx = -\frac{1}{(n+1)^2}$$

Thus $\int_0^1 \frac{\ln x}{(1+x)}\,dx = -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

We have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + -$$

$$= \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + - \right)$$

$$- 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdot \right)$$

$$= \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + - \right)$$

$$- 2 \cdot \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

$$= \frac{\pi^2}{6} - \frac{1}{2}\frac{\pi^2}{6} = \frac{\pi^2}{12}$$

$$\therefore \int_0^1 \frac{\ln x}{x+1}\,dx = -\frac{\pi^2}{12}.$$

Example

$$I = \int_0^{\infty} \frac{x e^{-x}}{e^x+1}\,dx$$

$$\frac{1}{e^x+1} = \frac{1}{e^x(1+e^{-x})} = e^{-x}(1 - e^{-x} + e^{-2x} - e^{-3x} + -)$$

So $I = \int_0^{\infty} x e^{-2x} \sum_{n=0}^{\infty} (-1)^n e^{-nx}\,dx$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^\infty x e^{-(n+2)x} \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\cancel{\left[\frac{x e^{-(n+2)x}}{n+2} \right]_0^\infty}^{\,0} + \frac{1}{n+2} \int_0^\infty e^{-(n+2)x} \, dx \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)^2} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n^2}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 1 - \frac{\pi^2}{12}.$$

<u>Example</u> $\tan^{-1} x = \int_0^x \frac{dy}{1+y^2}$

$$= \int_0^x (1 - y^2 + y^4 - y^6 + - \;) \, dy$$

$$|y|<1$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - -$$

$$\therefore \int_0^1 \frac{\tan^{-1} x}{x} \, dx$$

$$= \int_0^1 \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + - \right) dx$$

$$= \left[x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + - \right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

$$= G$$

G is Catalan's constant

$G = 0.9159 \ldots$