Properties of Convergent Sequences

Theorem Let x_n, y_n be convergent sequences with limits x, y respectively. Let $a, b \in \mathbb{R}$. Then

(1) $ax_n + by_n \to ax + by$

(2) $x_n y_n \to xy$.

(3) If $y_n \neq 0$ all n and $y \neq 0$, then

$$\frac{x_n}{y_n} \to \frac{x}{y}.$$

Proof (1) First suppose $a, b \neq 0$. If $a, b = 0$ then (1) says $0 = 0$. So it is trivial.

Recall that $x_n \to x$ if for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that if $n \geq N$

$$|x_n - x| < \varepsilon.$$

Another way of thinking of this is that the sequence $\{x_n\}$ eventually enters the interval $(x-\varepsilon, x+\varepsilon)$ and never leaves (for every ε)

For (1) write

$$\begin{aligned}
&|ax_n + by_n - ax - by| \\
&= |a(x_n - x) + b(y_n - y)| \\
&\leq |a(x_n - x)| + |b(y_n - y)| \\
&= |a||x_n - x| + |b||y_n - y|
\end{aligned}$$

Let $\varepsilon > 0$. Choose N_1 such that $n \geq N_1$, then $|x_n - x| < \frac{\varepsilon}{2|a|}$

Choose N_2 such that $n \geq N_2$

$$\Rightarrow |y_n - y| < \frac{\varepsilon}{2|b|}.$$

Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$

$$|ax_n + by_n - ax - by| < \frac{|a|\varepsilon}{2|a|} + \frac{|b|\varepsilon}{2|b|} = \varepsilon$$

So

$$ax_n + by_n \to ax + by.$$

For 2. we have the fact that if $\{x_n\}$ is convergent it is <u>bounded</u> ie There exists $M>0$ such that

$$|x_n| \leq M \text{ for all } n>0.$$

Let $\varepsilon>0$. Then consider

$$|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy|$$

$$= |y_n(x_n - x) + x(y_n - y)|$$

$$\leq |y_n(x_n-x)| + |x(y_n-y)|$$

Case 1 $x=0$. (Exercise). Case (2). Suppose $x \neq 0$. Then

$$|x_n y_n - xy| \leq |y_n||x_n - x| + |x||y_n - y|$$

$\{y_n\}$ is convergent so there is an M such that $|y_n| \leq M > 0$

$$\therefore |x_n y_n - xy| \leq M|x_n - x| + |x||y_n - y|$$

Choose N_1 s.t. $n \geq N_1 \Rightarrow |x_n - x| < \dfrac{\varepsilon}{2M}$

Choose N_2 st $n \geq N_2 \Rightarrow |y_n - y| < \dfrac{\varepsilon}{2|x|}$.

Let $N = \max\{N_1, N_2\}$. Then

$$n \geq N \Rightarrow |x_n y_n - xy| < M\frac{\varepsilon}{2M} + |x|\frac{\varepsilon}{2|x|} = \varepsilon.$$

So $x_n y_n \to xy$.

(3) If $y_n \neq 0$, $y \neq 0$ $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$.

Let $\varepsilon>0$ Consider

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| = \left|\frac{x_n y - x y_n}{y_n y}\right|$$

<u>Exercise</u> $|y_n y|$ is bounded. Add and subtract etc.

Theorem 1.20 Every monotone increasing sequence which is bounded above has a limit. Similarly every monotone decreasing sequence which is bounded below has a limit.

Proof Consider the set $\{x_1, x_2, \ldots, x_n, \ldots\}$ where $\{x_n\}$ is our sequence. This is bounded above by assumption. So there is a least upper bound. Call it x.

Pick $\varepsilon > 0$ and choose N s.t.

$x_N > x - \varepsilon$. We can do this because x_n is increasing so eventually it will pass any number which is not the least upper bound x.

$\therefore |x_n - x| = x - x_n < \varepsilon$ for all $n \geq N$.

$\therefore x_n \to x$.

The other case is similar.

Definition Let $\{x_n\}$ be a sequence. A subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ is formed by choosing a sequence of natural numbers $n_k \to \infty$

Example (1) $x_n = (-1)^n$

Let $n_k = 2k$

$\{x_{2k}\} = \{1, 1, 1, \ldots\}$

(2) $x_n = \dfrac{1}{n^2+1}$, $n_k = 3k$

$$x_{3k} = \frac{1}{9k^2+1}$$

Theorem Every sequence has a monotone sequence.

Theorem (Bolzano–Weierstrass). Every bounded sequence of real numbers has a convergent subsequence.

Proof If $\{x_n\}$ is bounded ~~any~~ so it has a monotone subsequence bounded above (or below) which must converge by our previous result.

Definition A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $n, m \geq N$ then $|x_n - x_m| < \varepsilon$.

Consider $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $n = 1, 2 \ldots$

Does this converge? Yes. We show that it is Cauchy. This is on a tutorial.

Proposition Every convergent sequence is a Cauchy sequence.

Proof Let $\varepsilon > 0$, and $x_n \to x$

Then
$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m|$$

Choose N_1 st. $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2}$

" N_2 " $n \geq N_2 \Rightarrow |x - x_m| < \varepsilon/2$

Let $n \geq N = \max\{N_1, N_2\}$. Then for $n \geq N$
$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{x_n\}$ is Cauchy.

The converse is true. It is harder to prove

<u>Proposition</u> If a ~~Con~~ Cauchy sequence has a convergent subsequence with limit x, then the Cauchy sequence converges to x.

<u>Proof</u> Suppose $\{x_n\}_{n=1}^{\infty}$ is Cauchy and there is a subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x$ as $n_k \to \infty$. Let $\varepsilon > 0$. Choose N large enough to make $|x_n - x_m| < \frac{\varepsilon}{2}$ for all $n, m \geq N$. Choose k large enough to make $n_k > N$.

So $|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x|$

$\leq |x_n - x_{n_k}| + |x_{n_k} - x|$

If $n, n_k > N$ $\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore$ x_n converges to x.

<u>Theorem</u> Every Cauchy sequence is convergent

<u>Proof</u> First every Cauchy sequence is bounded. (Exercise). BW says there is a convergent subsequence and so by previous result, the Cauchy sequence converges.

Consider an example.

Let $x_n = \sum_{k=1}^{n} \frac{1}{k}$

$$x_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} = x_n + \frac{1}{n+1}$$

$$|x_{n+1} - x_n| = \frac{1}{n+1} \to 0 \text{ as } n \to \infty$$

$H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number

$H_n \to \infty$ as $n \to \infty$. However
$\lim_{n\to\infty} (H_n - \ln n) = \gamma$ Euler's constant

$\gamma \approx .5667..$ It is not known if it is rational or irrational.

Now
$$\sum_{k=n}^{2n} \frac{1}{k} = \frac{1}{n} + \cdots + \frac{1}{2n}$$
$$\geq n \times \frac{1}{2n} = \frac{1}{2}$$

$$\sum_{n=2n+1}^{4n} \frac{1}{k} = \frac{1}{2n+1} + \cdots + \frac{1}{4n}$$
$$\geq 2n \cdot \frac{1}{4n} = \frac{1}{2}$$

$$\sum_{n=4n+1}^{8n} \frac{1}{4k} = \frac{1}{4n+1} + \cdots + \frac{1}{8n}$$
$$\geq 4n \cdot \frac{1}{8n} = \frac{1}{2} \quad \text{etc}$$

So $\sum_{n=1}^{\infty} \frac{1}{k} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

$\therefore$ $H_n \to \infty$ as $n \to \infty$. So $\{H_n\}$ is not <u>Cauchy</u>.