Series We turn now to series. A series is the sum of a sequence a_n. We can have finite series such as

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

Or we can have infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Infinite series arise in the solution of many problems. The most important question we can ask about an infinite series is whether or not it converges? We define convergence in terms of sequences.

Definition Let $S_N = \sum_{n=1}^{N} a_n$, where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. If $\{S_N\}_{N=1}^{\infty}$ converges to a limit S, then we say that $\sum_{n=1}^{\infty} a_n$ is a convergent series and

$$\sum_{n=1}^{\infty} a_n = S.$$

If $\sum_{n=1}^{\infty} a_n$ does not converge it diverges

Ex Let $\{p_n\}$ be the sequence of all primes. Consider

$$\sum_{n \geq 1} \frac{1}{p_n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty$$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{p_n} < \infty$. This is by the alternating series test

$\sum_{n=1}^{\infty} \frac{(-1)^n n}{p_n} = ?$ it is not known if this converges

Theorem If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are convergent with sums S and T, then $\sum_{n=1}^{\infty}(a_n+b_n) = S+T$

Also $\sum_{n=1}^{\infty} c\, a_n = cS$, $-\infty < c < \infty$

Proof $S_N = \sum_{n=1}^{N} a_n$, converges $T_N = \sum_{n=1}^{N} b_n$ converges. So $S_N + T_N$ converges
Hence $\sum_{n=1}^{\infty}(a_n+b_n) = S+T$.

Definition A series is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, $\sum_{n=1}^{\infty} a_n$ is conditionally convergent

Example $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

But $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} -$
$= \ln 2$.

This is important

Lemma If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Proof $S_n = \sum_{k=1}^{n} a_k$ and $\{S_n\}_{n=1}^{\infty}$ is convergent with limit S. Then $S_{n+1} - S_n \to 0$
But $S_{n+1} - S_n = a_n$.

<u>Propositition</u> Let $\sum_{n=1}^{\infty} a_n$ be convergent. Then as $N, M \to \infty$ $(N > M)$ $\sum_{n=M+1}^{N} a_n \to 0$.

i.e. $a_{M+1} + \cdots + a_N \to 0$ as $N, M \to \infty$

<u>Proof</u> (Exercise)

<u>Proposition</u> If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} b_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent

<u>Proof</u> Consider $S_N = \sum_{n=1}^{N} a_n b_n$. Let $\sum_{n=1}^{\infty} a_n$ be convergent. So $\lim_{n \to \infty} a_n = 0$

$\{a_n\}$ is convergent, so it is bounded

Let $|a_n| \le K$, $0 < K < \infty$. Suppose $\sum_{n=1}^{\infty} |b_n| < \infty$. If $N > M$

$$|S_N - S_M| = \left|\sum_{n=M+1}^{N} a_n b_n\right| \le \sum_{n=M+1}^{N} |a_n b_n|$$

$\left(S_N = \sum_{n=1}^{M} + \sum_{n=M+1}^{N}\right.$. So $\left.S_N - S_M = \cancel{\sum_{n=1}^{M}} + \sum_{n=M+1}^{N} - \cancel{\sum_{n=1}^{M}}\right)$

$$\therefore \quad |S_N - S_M| \le \sum_{n=M+1}^{N} |a_n b_n|$$

$$\le K \sum_{n=M+1}^{N} |b_n| \to 0$$ by previous result.

So $\{S_N\}$ is Cauchy. Every Cauchy sequence converges, so $\sum_{n=1}^{\infty} a_n b_n$

$\therefore$ converges

If $|r|<1$, the series

$$S = a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

The geometric series

Take $a=1$, $r=-x^2$, $|x|<1$

Then

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

This is a Taylor series.

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad |x|<1$$

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

__Example__ $S = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. $\qquad \left(\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}\right)$

__Proof__

$$\begin{aligned} S_n &= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \\ &< 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \frac{1}{n(n-1)} \\ &= 1 + \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \frac{1}{n-1} - \frac{1}{n} \\ &= 2 - \frac{1}{n} \to 2 \text{ as } n \to \infty \end{aligned}$$

$\therefore$ So $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$

In fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ ← This will be useful later.

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

$\sum_{n=1}^{\infty} \frac{1}{n^3} = ?$ In fact $\sum_{n=1}^{\infty} \frac{1}{n^{2k+1}}$ are not known

<u>Theorem</u> (Comparison Test) Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series of positive terms. If there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent. Conversely if $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent

<u>Proof</u> Let $T = \sum_{k=1}^{\infty} b_k$, $S_n = \sum_{k=1}^{n} a_k$

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_{N-1} + a_N + \cdots + a_n \\ &\leq a_1 + \cdots + a_{N-1} + b_N + \cdots + b_n \\ &= b_1 + \cdots + b_{N-1} + b_N + \cdots + b_n \\ &\quad + (a_1 - b_1) + \cdots + (b_{N-1} - a_N) \\ &= T_n + \sum_{k=1}^{N-1} (a_k - b_k) \end{aligned}$$

$$T_n = \sum_{k=1}^{n} b_n \longrightarrow T.$$

So S_n monotone increasing and bounded above by $T + \sum_{k=1}^{N-1} (a_k - b_k)$.

$\therefore \{S_n\}$ is convergent

The proof of the converse is similar

Example (1) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Now $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n}$ Because

$\sqrt{n} \leq n$ all $n \geq 1$

So $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$. Hence $\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{N} \frac{1}{n}$.

$\sum_{n=1}^{N} \frac{1}{n} \to \infty$, So $\sum_{n=1}^{N} \frac{1}{\sqrt{n}} \to \infty$ as $N \to \infty$.

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. For $n \geq 1$, $n^2 \leq n^6$

Thus $\frac{1}{n^6} \leq \frac{1}{n^2}$. Hence

$$\sum_{n=1}^{N} \frac{1}{n^6} \leq \sum_{n=1}^{N} \frac{1}{n^2} \to \frac{\pi^2}{6}$$

So $\sum_{n=1}^{N} \frac{1}{n^6}$ converges

Theorem (The limit comparison test)
Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series of strictly positive terms. Suppose that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = l \neq 0$$

Then either both series converge or both series diverge

Proof Notes

Example We know $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

Consider $\sum_{n=1}^{\infty} \frac{n+1}{2n^3+n+3}$

$$\lim_{n \to \infty} \frac{n+1}{2n^3+n+3} \Big/ \frac{1}{n^2} = \lim_{n \to \infty} \frac{n^3+n^2}{2n^3+n+3}$$

$$= \frac{\lim_{n\to\infty} \cancel{n^3}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty} \cancel{n^3}\left(2+\frac{1}{n^2}+\frac{3}{n^3}\right)}$$

$$= \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty} 2+\frac{1}{n^2}+\frac{3}{n^3}} = \frac{1}{2} \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges so $\sum_{n=1}^{\infty} \frac{n+1}{2n^3+n+3}$ also converges

__Theorem__ (The ratio test) Let $\sum_{n=1}^{\infty} a_n$ be a series of strictly positive terms. Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. The series converges if $L<1$, diverges if $L>1$ and is inconclusive if $L=1$.

__Example__ $\sum_{n=1}^{\infty} \frac{n}{e^n}$ Here $a_n = \frac{n}{e^n}$

$$a_{n+1} = \frac{n+1}{e^{n+1}}, \quad a_n = \frac{n}{e^n}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{n+1}{e^{n+1}} \Big/ \left(\frac{n}{e^n}\right)$$

$$= \frac{1}{e}\left(\frac{n+1}{n}\right) = \frac{1}{e}\left(1+\frac{1}{n}\right) \to \frac{1}{e} < 1$$

$\therefore$ Series converges by the ratio test.