Calculus Part One Differentiation

We start with the famous definition

Definition 3.1 A function $f: X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$ is open, is said to be differentiable at x if

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (A)$$

exists. We say that $f'(x)$ is the derivative of f at x. We also write $f'(x) = \frac{df}{dx}$.

Equivalently

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad (B)$$

(A) and (B) are both called the Newton quotient

Example The most important case is $f(x) = x^n$. Let $n \in \{1, 2, \dots\}$

From the Binomial Theorem for $n > 1$

$$(x+h)^n = x^n + nx^{n-1}h + n(n-1)h^2x^{n-2} + \dots + h^n$$

So

$$\frac{f(x+h) - f(x)}{h} = \frac{x^n + nx^{n-1}h + n\frac{(n-1)}{2}h^2x^{n-2} + \dots + h^n - x^n}{h}$$

$$= nx^{n-1} + \frac{n(n-1)}{2}hx^{n-2} + \dots + h^{n-1}$$

So

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0}\left(nx^{n-1} + \frac{n(n-1)}{2}hx^{n-2} + \dots + h^{n-1}\right)$$

$$= nx^{n-1}.$$

If $n = 1$ $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = 1$

So $\frac{d}{dx} x = 1$.

If $f(x) = c$ a constant $f'(x) = 0$

Theorem Suppose that $f'(a)$ exists. Then f is continuous at a.

Proof Suppose $f'(a)$ exists

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f(x) = f(a) + \left(\frac{f(x) - f(a)}{x - a}\right)(x - a)$$

Thus

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} f(a) + \lim_{x \to a} (x-a)\left(\frac{f(x) - f(a)}{x - a}\right) \\
&= f(a) + \lim_{x \to a}(x-a) \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right) \\
&= f(a) + 0 \times f'(a) \\
&= f(a)
\end{aligned}$$

So f is continuous at $x = a$, because it satisfies the defn of continuity.

Theorem Let c be a constant and let f, g be differentiable at x_0. Then

(i) $(cf)'(x_0) = c f'(x_0)$

(ii) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

(iii) $(fg)'(x_0) = f'(x_0) g(x_0) + f(x_0) g'(x_0)$

Proof We do (iii) The product rule

$$\begin{aligned}
(fg)'(x_0) &= \lim_{x \to x_0} \frac{f(x) g(x) - f(x_0) g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{f(x) g(x) - f(x_0) g(x) + f(x_0) g(x) - f(x_0) g(x_0)}{x - x_0} \\
&= \lim_{x \to x_0} \left[g(x)\left(\frac{f(x) - f(x_0)}{x - x_0}\right) + f(x_0)\left(\frac{g(x) - g(x_0)}{x - x_0}\right)\right]
\end{aligned}$$

$= g(x_0)f'(x_0) + f(x_0)g'(x_0).$

The chain rule Suppose that g is differentiable at x and f is differentiable at $y = g(x)$. Then

$$\frac{d}{dx}(f\circ g)(x) = \frac{d}{dx}(f(g(x))) = f'(y)g'(x)$$

Proof Write $k = g(x+h) - g(x)$. g is diffble at x, so it is continuous there

So as $h \to 0$ $k = g(x+h) - g(x) \to 0$

Now

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \, \frac{g(x+h) - g(x)}{h}$$

$$= \frac{f(y+k) - f(y)}{k} \, \frac{g(x+h) - g(x)}{h}$$

$$\to f'(y)g'(x)$$

as $h \to 0$. The rest of the proof is in notes

Corollary What is $f(x) = \frac{1}{g(x)}$

First $\frac{d}{du}\frac{1}{u} = -\frac{1}{u^2}$, Let $u = g$

Then $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{1}{u^2}u'$

$$= -\frac{g'(x)}{(g(x))^2}$$

Now

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)}{g(x)} + f(x)\frac{d}{dx}\left(\frac{1}{g(x)}\right)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$ — The quotient rule

Now we cheat and use the integral before we have defined it. We know

$\ln(e^x) = x$ and $e^{\ln x} = x$. However we can define

$$\ln y = \int_1^y \frac{dt}{t}$$

It then follows that $\frac{d}{dy}\ln y = \frac{1}{y}$

What about e^x. If $f(x) = e^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= e^x \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{=1}$$

But how do we show this?

$\lim_{n \to \infty} \left(1 + \frac{h}{n}\right)^n = e^h$. By the Binomial Theorem

$$\frac{1}{h}\left(\left(1 + \frac{h}{n}\right)^n - 1\right) = 1 + \sum_{k=2}^{n} \binom{n}{k} \frac{h^{k-1}}{n^k}$$

Let $g(h) = \sum_{k=2}^{\infty} \binom{n}{k} \frac{h^{k-2}}{n^k}$. This converges by the ratio test

$$\lim_{h\to 0}\frac{e^h-1}{h} = \lim_{h\to 0}\lim_{n\to\infty}\left(1+\sum_{k=2}^{n}\binom{n}{k}\frac{h^{k-1}}{n^k}\right)$$

$$= \lim_{h\to 0}\left(1+hg(h)\right) = 1$$

$$\therefore \frac{d}{dx}e^x = e^x$$

Example $f(x) = \sin x$

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h}$$

$$= \frac{\sin x\cosh + \sin h\cos x - \sin x}{h}$$

$$= \sin x\frac{(\cos h-1)}{h} + \cos x\,\frac{\sin h}{h}$$

Now $\lim_{h\to 0}\frac{\sin h}{h} = 1$ (Proof uses Trig)

$$\lim_{h\to 0}\frac{\cos h-1}{h} = 0.$$

$$\therefore f'(x) = \cos x$$

Example $f(x) = \tan^{-1}x$. We know

$$\frac{d}{dx}\tan y = \frac{d}{dy}\left(\frac{\sin y}{\cos y}\right) = \frac{\cos^2 y+\sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$

$$= \sec^2 y.$$

Now $\tan^{-1}(\tan x) = x$. Let $y = \tan x$

$$\frac{d}{dx}\tan^{-1}(\tan x) = \frac{d}{dy}(\tan^{-1}y)\frac{dy}{dx} = \frac{d}{dx}x = 1$$

$$\therefore \frac{d}{dy}(\tan^{-1}y)\sec^2 x = 1$$

$$\therefore \frac{d}{dy}\tan^{-1}y = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x}$$

$$= \frac{1}{1+y^2}$$

We can calculate higher derivatives in the obvious way

$$\frac{d^2f}{dx^2} = \frac{d}{dx} f'(x) \quad , \quad \frac{d^3f}{dx^3} = \frac{d}{dx}\frac{d^2f}{dx^2}$$

etc

Partial derivatives Let $f: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

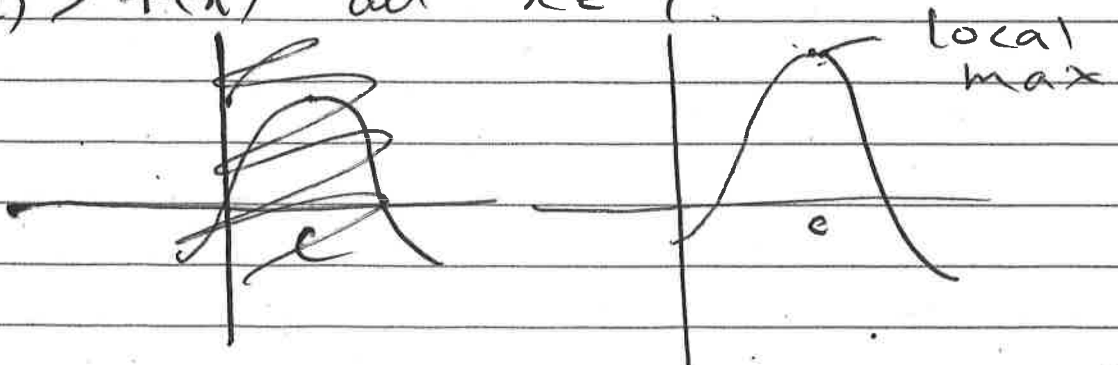
$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

The most important fact is that if f is continuous with continuous derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Maxima and Minima We start with a definition.

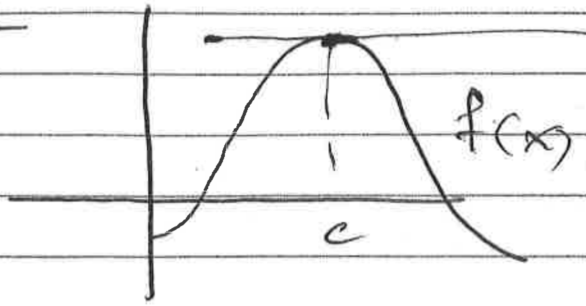
Definition A function $f: X \to \mathbb{R}$ has a local maximum at $c \in X$ if there is a subset $Y \subseteq X$, such that $c \in Y$ and $f(c) > f(x)$ all $x \in Y$.



similarly c is a local minimum if $f(c) < f(x)$ all $x \in Y$. c is called a maximiser or a minimiser or an extreme point.

Theorem 3.6 Let I be an open interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ is differentiable at $c \in I$. If f attains a local maximum or minimum at c, then $f'(c) = 0$

Proof



tangent has zero gradient at c
$\therefore f'(c) = 0$

We proceed by contradiction. Assume $f'(c) > 0$. Choose $\delta > 0$ such that for $0 < |x-c| < \delta$

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c)$$

Pick $x > c$, with $|x - c| < \delta$. Then we have

$$-f'(c) < \frac{f(x) - f(c)}{x - c} - f'(c) < f'(c)$$

$$\therefore \frac{f(x) - f(c)}{x - c} > 0 \qquad \therefore f(x) > f(c).$$

This is impossible as $f(c)$ is a local maximum. So $f'(c)$ cannot be positive. Let $f'(c) < 0$.

Pick $\delta > 0$, $x \in I$ st $0 < |x - c| < \delta$ and

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < -f'(c)$$

Then with $x < c$

$$f'(c) < \frac{f(x) - f(c)}{x - c} - f'(c) < -f'(c)$$

or $\dfrac{f(x) - f(c)}{x - c} < 0$ but $x < c$

$f(x) > f(c)$.

Contradiction $\therefore f'(c) = 0$

<u>Rolle's Theorem</u> Let $[a,b]$ be a closed interval in $\mathbb{R}$. Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$

<u>Proof</u> f has a local maximum and minimum in $[a,b]$. Let $c \in (a,b)$ be a max or a min $\therefore f'(c) = 0$
If max of f is at a and at b then $f(a) = f(b)$ so f is const

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<u>Mean Value Theorem</u> Let $[a,b]$ be a closed and bounded interval on $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b). Then there is a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

<u>Proof</u> Let $g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b-a}\right)(x-a)$

This is differentiable and

$$g(a) = f(a) - f(a) - 0 = 0$$

$$g(b) = f(b) - f(a) - \left(\frac{f(b) - f(a)}{b-a}\right)(b-a)$$

$$= f(b) - f(a) - f(b) + f(a) = 0$$

$\therefore g(a) = g(b) = 0 \therefore \exists c$ st

$$g'(c) = f'(c) - \left(\frac{f(b) - f(a)}{b-a}\right) = 0$$

by Rolle's Theorem

If $|f'(x)| \leq M$ on $[a,b]$.

$$\left|\frac{f(b)-f(a)}{b-a}\right| = |f'(c)| \quad \text{for some } c \in (a,b)$$

$$\text{So } |f(b)-f(a)| \leq M|b-a|$$

So f is Lipschitz continuous

Additional note Let $f(x) = x$ on $[0,1]$ the maximum is at $x=1$ what about $f'(1)$? In fact $f'(1)$ does not exist. The derivative is only defined on $(0,1)$ so the derivative test does not apply here.