<u>Taylor's Theorem</u> Let I be an open interval in $\mathbb{R}$, $n \in \mathbb{N}$, $f \in C^{n+1}(I)$. Let $a \in I$ and $x \in I$, $x \neq a$. Then there is a point ξ between a and x such that

$$f(x) = f(a) + f'(x)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$ (last term is the error)

Note $f \in C^{n+1}(I)$ means f is $n+1$ times differentiable on I

<u>Proof</u> As with the mean value theorems, we use Rolle's Theorem. The key is to construct the right function. We set

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{1}{2!} f''(t)(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n$$

Observe that $F(x) = f(x) - f(x) - f'(x)(x-x) - \cdots = 0$

F is differentiable, so we calculate

$$F'(t) = -f'(t) - f''(t)(x-t) + f'(t) - \frac{1}{2} f'''(t)(x-t)^2 + f''(t)(x-t) + \cdots - \frac{n f^{(n)}(t)(x-t)^{n-1}}{n!} - \frac{f^{(n+1)}(t)(x-t)^n}{n!} + \frac{n f^{(n)}(t)(x-t)^{n-1}}{n!}$$

$$= -\frac{f^{(n+1)}(t)(x-t)^n}{n!}.$$

Let

$$G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a)$$

$$G(a) = F(a) - \left(\frac{x-a}{x-a}\right)^{n+1} F(a) = 0$$

$$G(x) = F(x) - \left(\frac{x-x}{x-a}\right)^{n+1} F(a) = F(x) = 0$$

G is differentiable, So there is an ξ between a and x such that

$G'(\xi) = 0$, This is by Rolle's Theorem, Now

$$G'(t) = F'(t) + (n+1)\frac{(x-t)^n}{(x-a)^{n+1}} F(a)$$

$$\therefore \quad F'(\xi) + (n+1)\frac{(x-\xi)^n}{(x-a)^{n+1}} F(a) = 0$$

$$F'(\xi) = -\frac{f^{(n+1)}(\xi)(x-\xi)^n}{n!}$$

$$\therefore \quad (n+1)\frac{\cancel{(x-\xi)^n}}{(x-a)^{n+1}} F(a) = f^{(n+1)}(\xi)\frac{\cancel{(x-\xi)^n}}{n!}$$

$$F(a) = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}$$

But

$$F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{1}{2}f''(t)(x-t)^2 \dots - \frac{f^{(n)}(t)(x-t)^n}{n!}$$

$$\therefore \quad F(a) = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!} = f(x) - f(a) - f'(a)(x-a) \dots - \frac{f^{(n)}(a)(x-a)^n}{n!}$$

$$\therefore \quad f(x) = f(a) + f'(a)(x-a) + \dots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-a)^{n+1}$$

If $\dfrac{f^{(n+1)}(\xi)}{(n+1)!} \to 0$ as $n \to \infty$ we say that f is equal to its Taylor series. Such functions are said to be analytic:

Example $f(x) = \cos x$. Observe

$$\frac{|f^{(n+1)}(\xi)|}{n+1} \leq \frac{1}{(n+1)!}$$

Since $\dfrac{d^{n+1}}{dx^{n+1}} \cos x$ is $\pm \sin x$ or $\pm \cos x$

$$\therefore \lim_{n \to \infty} \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right| = 0$$

So as n increases the error in the Taylor polynomial $\to 0$. So for every x

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Similarly

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$f(x) = e^x$ $\quad \dfrac{f^{(n+1)}(\xi)}{(n+1)!} = \dfrac{e^{\xi}}{(n+1)!}$, e^{ξ} is a constant

$$\text{So } \frac{e^{\xi}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

$$\therefore e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$= \cos x + i \sin x$. - Euler's Formula

[$ix = \ln(\cos x + i \sin x)$ was proved before Euler by Roger Coates, who did nothing with it]

<u>Theorem</u> e is an irrational number.

<u>Proof</u> we know that

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Suppose $e = \frac{a}{b}$, a, b are positive integers

Since e is not an integer $b > 1$

Pick $N > b$.

Then $N! = 1 \times 2 \times \cdots \times b \times (b+1) \cdots \times N$

Thus $N! e = 1 \times 2 \times \cdots \times \cancel{b} \times \cdots \times N \times \frac{a}{\cancel{b}}$

$= \text{integer} \times a = \text{integer}.$

$e = \sum_{n=0}^{N} \frac{1}{n!} + \sum_{n=N+1}^{\infty} \frac{1}{n!}$. So we have

$$N! e = \sum_{n=0}^{N} \frac{N!}{n!} + \sum_{n=N+1}^{\infty} \frac{N!}{n!}$$

Now if $n \le N$ $\frac{N!}{n!}$ is an integer

So $\sum_{n=0}^{N} \frac{N!}{n!}$ is an integer.

$N!e - \sum_{n=0}^{N} \frac{N!}{n!}$ is an integer. Call it M

Clearly $M > 0$, since $N!e > \sum_{n=0}^{N} \frac{N!}{n!}$.

So $M = \sum_{n=N+1}^{\infty} \frac{N!}{n!}$

$$= \frac{1}{N+1} + \frac{1}{(N+1)(N+2)} + \frac{1}{(N+1)(N+2)(N+3)} + \cdots$$

$$< \frac{1}{N+1} + \frac{1}{(N+1)^2} + \frac{1}{(N+1)^3} + \cdots$$

$$= \frac{\frac{1}{N+1}}{1 - \frac{1}{N+1}} = \frac{1}{\cancel{N+1}} \frac{\cancel{N+1}}{(N+1) - 1}$$

$= \frac{1}{N} < 1$, but this is <u>not</u> an integer.

So we have a contradiction.

Thus e is not rational.

<u>Infinite Products</u> Euler introduced the concept of an infinite product expansion for a function

<u>Definition</u>

$$\prod_{k=1}^{n} a_k = a_1 \times a_2 \times \cdots \times a_n$$

is a finite product. For an infinite product

$$\prod_{k=1}^{\infty} a_k = a_1 \times a_2 \times \cdots \times a_n \times \cdots$$

$$= \lim_{n \to \infty} \prod_{k=1}^{n} a_k$$

Euler showed $\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right)$

Problem What is $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

Euler proved

Theorem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof We know that

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$ infinite product

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$ Taylor series

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

Now

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right) = \left(1 - \frac{z^2}{\pi^2}\right)\left(1 - \frac{z^2}{4\pi^2}\right)\left(1 - \frac{z^2}{9\pi^2}\right) \cdots$$

$$= 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots\right) z^2$$

+ higher order terms

$$= 1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \frac{z^2}{\pi^2} + \text{higher terms}$$

$$\therefore \quad 1 - \frac{z^2}{6} + \frac{z^4}{5!} - \cdots = 1 - \left(\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\right) z^2$$

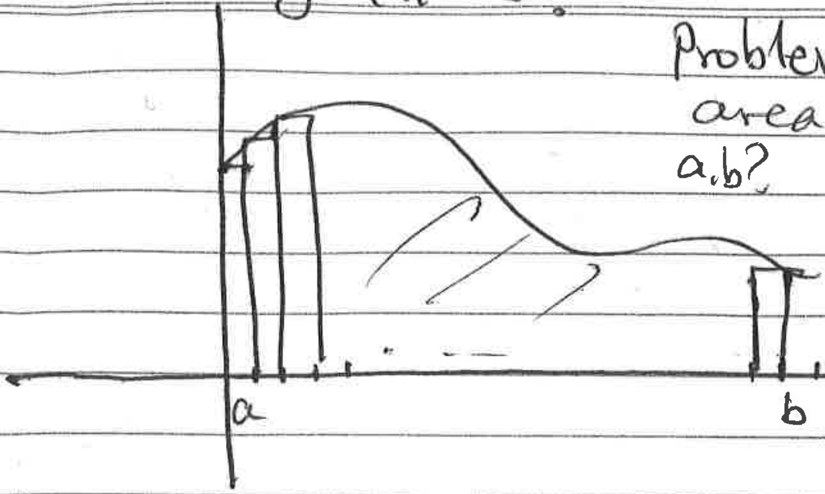
+ higher terms

So
$$-\frac{1}{6} = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

or
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Integration

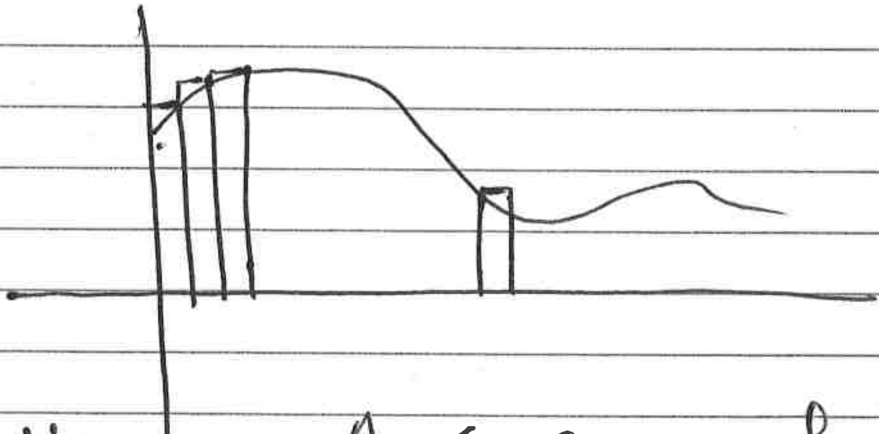
Problem what is the area under f between a,b?



The area A under the curve is approximated by the sum of the areas of the rectangles

A > area of the rectangles

We could also take rectangles lying above the curve



Here A < sum of areas of rectangles above curve.

Let $[a,b]$ be partitioned by
$P = \{x_0, x_1, \ldots, x_n\}$, $x_0 < x_1 < \ldots < x_n$
$[x_0, x_1], [x_1, x_2), [x_2, x_3) \ldots [x_{n-1}, x_n]$

Let f be continuous on $[a,b]$
Let $f(\bar{x}_i)$ be the minimum of f on $[x_i, x_{i+1})$

Let $f(x_i^*)$ be the maximum of f on $[x_i, x_{i+1})$

The lower Riemann sum is

$$L(f,P) = \sum_{i=1}^{n} f(\bar{x}_i)(x_i - x_{i-1})$$

The upper sum is

$$U(f,P) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

So $L(f,P) \leq A \leq U(f,P)$.

The idea is to add more and more rectangles and hopefully the Riemann sums will converge to the area under the curve. How do we make this precise?