Theorem Every Cauchy sequence is bounded.

Proof Choose N such that $|x_n - x_m| < 1$ for all $n, m \geq N$. Pick $m = N$, so that

$$|x_n - x_N| < 1 \quad \text{all } n \geq N.$$

Now $|a| - |b| \leq |a-b|$ (*).

So $|x_n| - |x_N| \leq |x_n - x_N| < 1$.

Hence $|x_n| < |x_N| + 1$ for all $n \geq N$.

Now choose

$$M = \max\{|x_1|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

Then for all n

$$|x_n| \leq M$$

So $\{x_n\}$ is bounded.

To prove $|a-b| \geq |a| - |b|$ (Which is very useful)

(*) We use the triangle inequality. We know that $|c+d| \leq |c| + |d|$.

Let $d = a-b$, $c = b$. Then

$$|b + a - b| \leq |b| + |a-b|$$

or

$$|a| - |b| \leq |a-b|.$$