Proof of a limit property from class

If $x_n \to x$ and $y_n \to y \neq 0$ and y_n is never zero, then

$$\frac{x_n}{y_n} \to \frac{x}{y}.$$

<u>Proof</u>. Let $\varepsilon > 0$. We have

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| = \left|\frac{x_n y - x y_n}{y_n y}\right|$$

$$= \frac{1}{|y_n y|}|x_n y - x y_n|$$

The add and subtract trick

$$= \frac{1}{|y_n y|}|x_n y - x_n y_n + x_n y_n - x y_n|$$

$$= \frac{1}{|y_n y|}|x_n(y - y_n) + y_n(x_n - x)|$$

Triangle inequality

$$\leq \frac{|x_n|}{|y_n y|}|y - y_n| + \frac{|y_n|}{|y_n y|}|x_n - x|$$

$$= \frac{|x_n|}{|y_n y|}|y - y_n| + \frac{1}{|y|}|x_n - x|$$

Now convergent sequences are bounded so there an $M > 0$, $M < \infty$ with

$$|x_n| \leq M.$$

Now $|y| \neq 0$ and we can find $N_1 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon |y|}{2}$,

(Remember we can make $|x_n - x|$ as small as we desire by choosing n large enough. This is the definition of convergence.

Now $\left|\frac{x_n}{y_n} - \frac{x}{y}\right| \leq \frac{M|y-y_n|}{|y_n y|} + \frac{1}{|y|}|x_n - x|$

Now if $|y-y_n| = |y_n - y| < a$

we have

$$y - a < y_n < y + a$$

by convergence of y_n we can make a anything we like. Let $a = \frac{1}{2}y$ then we can find N_2 such that $n \geq N_2 \Rightarrow y_n > \frac{1}{2}y$. Then for $n > N_2$

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| \leq \frac{M}{\frac{1}{2}|y|^2}|y-y_n| + \frac{1}{|y|}|x_n - x|$$

$$= \frac{2M}{|y|^2}|y-y_n| + \frac{1}{|y|}|x_n - x|$$

Choose N_3 such that $n \geq N_3$

$$\Rightarrow |y-y_n| < \frac{\varepsilon|y|^2}{4M}.$$

Let $N = \max\{N_1, N_2, N_3\}$ Then

$$n \geq N \Rightarrow \left|\frac{x_n}{y_n} - \frac{x}{y}\right| \leq \frac{2M}{|y|^2}|y-y_n| + \frac{1}{|y|}|x_n - x|$$

$$< \frac{2M}{|y|^2}\frac{\varepsilon|y^2|}{4M} + \frac{1}{|y|}\frac{\varepsilon|y|}{2}$$

$$= \varepsilon.$$

So $\frac{x_n}{y_n} \longrightarrow \frac{x}{y}$.

When we come to functions, you will see the same ideas and methods appearing again.