The Mean Value Theorem and the Fundamental Theorem of Calculus

Suppose that F is continuous on $[a,b]$ and $F' = f$ on (a,b)

Partition $[a,b]$ into $[x_0, x_1], [x_1, x_2] \ldots [x_{n-1}, x_n]$

$x_0 = a, \; x_n = b$

Then

$$F(b) - F(a) = F(x_n) - F(x_0)$$
$$= F(x_1) - F(x_0) + F(x_2) - F(x_1) + \cdots + F(x_n) - F(x_{n-1})$$

Now by the Mean value Theorem

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) = f(c_i) \qquad c_i \in [x_{i-1}, x_i]$$

Thus

$$F(b) - F(a) = \sum_{i=1}^{n} F'(c_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

The quantity on the right is a Riemann sum and as we let $\max|x_i - x_{i-1}| \to 0$

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \longrightarrow \int_a^b f(x)\,dx$$

Hence

$$\int_a^b f(x)\,dx = F(b) - F(a).$$