We wish to prove that $\frac{x_n}{y_n} \to \frac{x}{y}$. We assume that $y_n, y \neq 0$. Notice that $\{y_n\}$ is convergent so it is bounded. So there exists $M_2 > 0$ such that $|y_n| \le M_2$ all n. Now $y_n > 0$ and $y \neq 0$. So there is a lower bound for y_n which is positive ie. There is a $B > 0$ such that $y_n > B$ all n. If B were zero then some y_n would be zero, but this is not true by our condition on y_n. Thus for all n we have $\frac{1}{|y_n y|} \le \frac{1}{|y|B} < \infty$.

Now let $\varepsilon > 0$. Suppose $|x_n| \le M_1$ for some $M_1 < \infty$. ($|x_n|$ is also bounded)

Then

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| = \left|\frac{x_n y - x y_n}{y_n y}\right|$$

(add and subtract)

$$= \left|\frac{x_n y - x_n y_n + x_n y_n - x y_n}{y_n y}\right|$$

$$= \frac{1}{|y_n y|} \left| x_n (y - y_n) + (x_n - x) y_n \right|$$

Triangle inequality

$$\le \frac{1}{|y_n y|}\Big(|x_n| |y - y_n| + |y_n| |x_n - x| \Big)$$

$$\le \frac{M_1}{|y|B} |y - y_n| + \frac{M_2}{|y| B} |x_n - x| .$$

Choose $N_1 \in \mathbb{N}$ such that $n \ge N_1 \Rightarrow |y - y_n| < \frac{|y|B}{2M_1}\varepsilon$

" $N_2 \in \mathbb{N}$ " " $n \ge N_2 \Rightarrow |x_n - x| < \frac{|y|B}{2M_2}\varepsilon$

Then if $n \ge N = \max\{N_1, N_2\}$ we have

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| < \frac{M_1}{|y|B} \frac{|y|B}{M_1} \frac{\varepsilon}{2} + \frac{M_2}{|y|B} \frac{|y|B}{M_2} \frac{\varepsilon}{2} = \varepsilon .$$

So $\frac{x_n}{y_n} \to \frac{x}{y}$.