Real Analysis 35007.
Four Week Review

(i) Use the definition to prove that the sequence with nth term $a_n = \dfrac{4n+2}{3n+5}$ is convergent.

(ii) Explain why the sequence with nth term $a_n = \dfrac{(-1)^n n}{3n+2}$ has convergent subsequences. Find the lim inf and lim sup of the sequence.

(iii) Use properties of limits to determine $\lim_{n\to\infty} (3^n + 2^n)^{1/n}$.

(iv) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of positive terms and suppose $x_n \to l$ with $l > 0$. Prove that $\sqrt{x_n} \to \sqrt{l}$. Hint $|x_n - l|$ is a difference of two squares and $\frac{\sqrt{x_n}+\sqrt{l}}{\sqrt{x_n}+\sqrt{l}} = 1$. What happens if $l = 0$?

(v) From the definition, prove that $a_n = \dfrac{1}{n^2+4}$ is a Cauchy sequence.

(vi) Use the comparison test to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+4}$ is convergent.

Real Analysis 35007.
Solutions to Week Four Review Questions.

(i) We have $a_n = \dfrac{4n+2}{3n+5}$. We show $a_n \to \frac{4}{3}$. Let $\epsilon > 0$. We require

$$\left|\frac{4n+2}{3n+5} - \frac{4}{3}\right| = \left|\frac{3(4n+2) - 4(3n+5)}{3(3n+5)}\right| = \left|\frac{6-20}{3(3n+5)}\right|$$
$$= \frac{14}{3}\frac{1}{3n+5} < \epsilon.$$

So we require $\frac{1}{3n+5} < \frac{3\epsilon}{14}$ or $3n+5 > \frac{14}{3\epsilon}$. Take $N \in \mathbb{N}$ with $N > \frac{1}{3}\left(\frac{14}{3\epsilon} - 5\right)$. Then if $n \geq N$,

$$\left|\frac{4n+2}{3n+5} - \frac{4}{3}\right| < \epsilon.$$

So $a_n \to \frac{4}{3}$.

(ii) The sequence satisfies $\left|\frac{(-1)^n n}{3n+2}\right| < \frac{1}{3}$, so it is bounded. The Bolzano-Weierstrass Theorem says that it has a convergent subsequence. The lim sup cannot be larger than 1/3 and the lim inf cannot be smaller than $-1/3$. Take the subsequence a_{2k}. We have

$$a_{2k} = \frac{(-1)^{2k}2k}{6k+2} = \frac{k}{3k+1} \to \frac{1}{3}.$$

So $\limsup a_k = \frac{1}{3}$. Now take

$$a_{2k+1} = \frac{(-1)^{2k+1}(2k+1)}{6k+5} = \frac{-(2k+1)}{6k+5} \to -\frac{1}{3}.$$

So $\liminf a_k = -\frac{1}{3}$.

(iii) We have

$$(3^n + 2^n)^{1/n} = 3\left(1 + \left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}}.$$

Now $1 < \left(1 + \left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}} < (1+1)^{\frac{1}{n}} = 2^{\frac{1}{n}}$. Since $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} 2^{\frac{1}{n}} = 1$, by the squeeze (or sandwich) theorem (see Canvas notes), we have $\lim_{n\to\infty}\left(1 + \left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}} = 1$. So

$$(3^n + 2^n)^{\frac{1}{n}} \to 3.$$

(iv) We can write $|x_n - l| = |\sqrt{x_n} - \sqrt{l}||\sqrt{x_n} + \sqrt{l}|$. Now let $\epsilon > 0$. We can write

$$|\sqrt{x_n} - \sqrt{l}| = |\sqrt{x_n} - \sqrt{l}|\frac{|\sqrt{x_n} + \sqrt{l}|}{|\sqrt{x_n} + \sqrt{l}|}$$
$$= \frac{|x_n - l|}{|\sqrt{x_n} + \sqrt{l}|} \leq \frac{1}{\sqrt{l}}|x_n - l|.$$

Choose N such that if $n \geq N$, then $|x_n - l| < \sqrt{l}\epsilon$. Then $n \geq N$ implies

$$|\sqrt{x_n} - \sqrt{l}| < \frac{1}{\sqrt{l}}\sqrt{l}\epsilon = \epsilon.$$

So $\sqrt{x_n} \to \sqrt{l}$. If $l = 0$, then we choose N such that $n \geq N$ implies $x_n < \epsilon^2$. Which implies $|\sqrt{x_n} - 0| = |\sqrt{x_n}| < \epsilon$. So $\sqrt{x_n} \to 0$.

(1) (v) Suppose $n > m$. Let $\epsilon > 0$ By the triangle inequality

$$\left|\frac{1}{n^2+4} - \frac{1}{m^2+4}\right| \leq \frac{1}{n^2+4} + \frac{1}{m^2+4} \leq \frac{2}{m^2+4}.$$

Now $m^2 + 4 > m^2 > m$. So $\frac{2}{m^2+4} \leq \frac{2}{m}$. Choose $N > \frac{2}{\epsilon}$. Then $n, m \geq N$ implies

$$\left|\frac{1}{n^2+4} - \frac{1}{m^2+4}\right| \leq \frac{2}{m} < \epsilon.$$

So it is a Cauchy sequence.

(vi) . Clearly $n^3 + 4 > n^3$. We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Now for all N

$$\sum_{n=1}^{N} \frac{1}{n^3+4} < \sum_{n=1}^{N} \frac{1}{n^3}.$$

So the series converges by the comparison test.