A Taylor Expansion via Integration by parts.

We calculate a Taylor polynomial for e^x. Clearly

$$e^x - 1 = \int_0^x e^t\,dt = (t-x)e^t\Big]_0^x - \int_0^x (t-x)e^t\,dt. \text{ So we get}$$

$$e^x - 1 = x - \int_0^x (t-x)e^t\,dt$$

Note $\frac{d}{dt}x = 0$

$$\therefore e^x = 1 + x - \int_0^x (t-x)e^t\,dt$$

$$= 1 + x - \left[\frac{(t-x)^2}{2}e^t\right]_0^x + \int_0^x \frac{(t-x)^2}{2}e^t\,dt$$

$$= 1 + x + \frac{x^2}{2!} + \int_0^x \frac{(t-x)^2}{2}e^t\,dt$$

$$= 1 + x + \frac{x^2}{2} + \frac{(t-x)^3}{3!}e^t\Big]_0^x - \int_0^x \frac{(t-x)^3}{3!}e^t\,dt$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

In general

$$e^x = \sum_{k=0}^{N} \frac{x^k}{k!} - (-1)^N \int_0^x \frac{(t-x)^N}{N!}e^t\,dt$$

In fact as $N \to \infty$ $\left|\int_0^x \frac{(t-x)^N}{N!}e^t\,dt\right| \to 0.$

To see this note that

$$\left|\int_0^x \frac{(t-x)^N}{N!}e^t\,dt\right| \le e^x \left|\int_0^x \frac{(t-x)^N}{N!}\,dt\right|$$

$$= \frac{x^{N+1}}{(N+1)!}e^x \to 0 \text{ as } N \to \infty \text{ for}$$

each <u>fixed</u> x. This is by a tutorial exercise.

Thus $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. This is the familiar Taylor expansion.