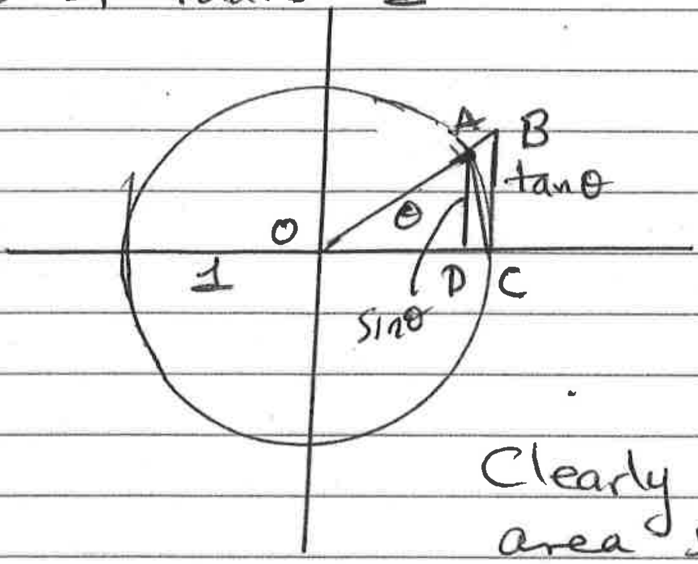
$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, where θ is in radians.
(This is not true if θ is in degrees)
We use some trig and geometry for this.
First note $\cos 0 = 1$.
We draw (badly) a circle of radius 1



(1) - The segment OAC has area $\frac{1}{2}\theta 1^2 = \frac{1}{2}\theta$.

(2) The length CB is $\tan\theta$.

(3) The length DA is $\sin\theta$.

Clearly triangle ODA has area smaller than segment OAC which has area smaller than OCB

The area of a triangle is $\frac{1}{2}$ base $\times$ height.

So $\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$, $\theta \in (0, \frac{\pi}{2})$

Or $\sin\theta < \theta < \tan\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, so this gives

$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$ or $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

which means $\cos\theta < \frac{\sin\theta}{\theta} < 1$.

Now $\cos\theta \to 1$ as $\theta \to 0$ so $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$. If $\theta \in (-\frac{\pi}{2}, 0)$ the proof is basically the same.

(2) $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$.

We deduce this from the previous limit.

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{(\cos\theta + 1)(\cos\theta - 1)}{\theta(\cos\theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}$$

$$= -\lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(\cos\theta + 1)}$$

$$= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \frac{\sin\theta}{\cos\theta + 1}$$

$$= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta + 1}$$

$$= -1 \times 0$$

$$= 0.$$