Introduction to Real Analysis.

Tutorial One.

Manipulations with inf and sup.

Recall that $\inf(A)$ is the greatest lower bound of the set of real numbers A and sup is the least upper bound of A.

(1) Suppose that $\xi > 0$ and S is a non empty set of real numbers bounded above. Prove that

$$\sup_{x \in S} \xi x = \xi \sup_{x \in S} x.$$

(2) Suppose that S is non empty, bounded above and that $S_0 \subseteq S$. (So S_0 is contained in S. It might equal S.) Prove that $\sup S_0 \leq \sup S$.

(3) Suppose that S is non empty, bounded above and that ξ is any real number. Prove that $\sup_{x \in S}(x + \xi) = \xi + \sup_{x \in S} x$.

(4) The distance between a point ξ and a set S is defined to be $d(\xi, S) = \inf_{x \in S} |\xi - x|$.

(a) If $\xi \in S$ prove that $d(\xi, S) = 0$.

(b) If S is bounded above and $\xi = \sup S$, prove that $d(\xi, S) = 0$. If S is bounded below, and $\xi = \inf S$ prove that $d(\xi, S) = 0$.

(c) If I is a closed interval, prove that $d(\xi, I) = 0$ implies that $\xi \in I$. If I is open, prove that we can always find an $\xi \notin I$ such that $d(\xi, I) = 0$.

Limits

Recall that a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a limit a as $n \to \infty$ is for every $\epsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Use the definition of a limit to establish the following.

(5) $\lim_{n \to \infty} \frac{n}{2n+4} = \frac{1}{2}$.

(6) $\lim_{n \to \infty} \frac{2n+1}{3n+2} = \frac{2}{3}$.

(7) $\lim_{n \to \infty} \frac{n}{n^2+1} = 0$.

Use properties of limits to show that

(8) $\lim_{n \to \infty} \left(\frac{2n^3 - 3n}{5n^3 + 4n^2 - 2} \right) = \frac{2}{5}$.

(9) $\lim_{n \to \infty} (\sqrt{n^2+4} - n) = 0$.

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Tutorial Two.

The starred problems are harder.

Understanding Limits.

(1) For what values of x does $\lim_{n\to\infty} \frac{x + x^n}{1 + x^n}$ exist?

(2) Suppose that $\{y_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $y_n \to 0$ as $n \to \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be another sequence of real numbers. Suppose that for all n, $|x_n - l| \leq y_n$. Prove that $x_n \to l$.

(3) Prove that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ converges. Note that if $a_1, ..., a_n$ are positive, then $(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{k=1}^{n} a_k$. (You will prove this later). Use your calculator to guess what the limit is.

(4) (*) Let $x > 0$ and let N be the smallest natural number such that $N > x$. Prove that

$$\frac{x^n}{n!} \leq \frac{x^{N-1}}{(N-1)!}\left(\frac{x}{N}\right)^{n-N+1}, \ n \geq N.$$

Conclude that $x^n/n! \to 0$ as $n \to \infty$. This result is essential for proving the convergence of power series.

(5) (*) Let α be any positive rational number and let $|x| < 1$. Show that there exists a natural number N such that

$$(1 + 1/N)^{\alpha+1}|x| \leq 1.$$

Deduce that

$$|n^{\alpha+1}x^n| \leq |N^{\alpha+1}x^N|,$$

for $n \geq N$. Hence show that $n^{\alpha}x^n \to 0$ as $n \to \infty$. This is also important in establishing the convergence of certain kinds of series.

Subsequences.

(6) Find a convergent subsequence of $\{\sin\left(\frac{\pi n}{2}\right)\}_{n=1}^{\infty}$.

(7) Suppose that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence and for any N, we can find $n \geq N$, such that $x_n \geq b$. Show that x_n has a subsequence which converges to a limit $l \geq b$.

(8) Find a convergent subsequence of $\left\{\frac{3^n + (-2)^n}{3^n - 2^n}\right\}_{n=1}^{\infty}$. What can you say about the limit in general?

(9) It can be shown that $n^{1/n} \to 1$ as $n \to \infty$. Suppose that we know that the limit exists, but we do not know its value. Determine the limit by considering the behaviour of the subsequence $\{(2n)^{\frac{1}{2n}}\}$.

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Tutorial Three.

lim sup and lim inf .

(1) Consider the sequence $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \ldots$. Determine the lim sup and lim inf for this sequence.

(2) Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence with limit superior given by l. Let the limit inferior be L. Show that for any $\epsilon > 0$ we can find $N > 0$ such that for all $n \geq N$, $x_n < l + \epsilon$. Formulate and prove the corresponding statement for x_n and L.

Cauchy Sequences.

(3) Suppose that $|x_{n+1} - x_n| \leq r^n$ where $0 < r < 1$. Prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

(4) We have a sequence defined by the recursive formula

$$x_{n+2} = (x_{n+1}x_n)^{1/2}.$$

Suppose that $0 < a \leq x_1 \leq x_2 \leq b$. Prove that $a \leq x_n \leq b$ for all $n \geq 0$. Hence establish the inequality

$$|x_{n+1} - x_n| \leq \frac{b}{a+b}|x_n - x_{n+1}|.$$

Deduce that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and hence converges.

(5) Let $x_1 = a, x_2 = b$. Set $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for all $n \geq 0$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges.

(6) How do calculators determine square roots? Most operations for determining function values are encoded into the hardware. Here is one algorithm. We let $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, $x_1 = x_0 > 0$, for $a > 0$. Prove that the sequence is convergent and that its limit is $\sqrt{a}$. Use this to obtain an approximation to the square root of 2. Hint: Let $y_n = \frac{x_n}{\sqrt{a}}$ and determine $\frac{y_n - 1}{y_n + 1}$. There are perfect squares involved.

The Bolzano-Weierstrass Theorem.

(7) Show that every point in the interval $[0, 1]$ is the limit of a subsequence of the sequence defined in Question one.

(8) (*) Given a set S of real numbers, let $S_\xi = \{x; x \in S, x \neq \xi\}$. We say that ξ is a limit point (or cluster point) of S if there is a sequence of points in S_ξ which converges to ξ. We can state a variation of the Bolzano-Weierstrass Theorem as follows. Every

bounded set with an infinite number of elements contains at least one limit point. Prove this.

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Tutorial Four.

Finite Sums.

(1) Evaluate the sum $\sum_{k=1}^{N} k$. Prove your result by induction.

(2) Find a formula for the sum $\sum_{k=1}^{N} k^2$. Prove your formula by induction.

(3) Sums of the form $\sum_{k=1}^{N} k^n$ can be shown to be given by polynomials in N of degree $n+1$. Use this to determine a formula for $\sum_{k=1}^{N} k^3$. You will need to solve a system of equations to find the coefficients of the fourth degree polynomial.

Infinite Sums.

(4) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+3}$ is convergent.

(5) Determine which of the following series converge and which diverge.

$$\text{(i)} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}, \ \text{(ii)} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}x^n, \ \text{(iii)} \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n},$$

$$\text{(iv)} \sum_{n=1}^{\infty} n^{\alpha} x^n, |x|<1, \alpha>0, \ \text{(v)} \sum_{n=1}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}.$$

(6) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)(n+5)} = \frac{23}{480}$. Hint: Use partial fractions.

(7) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)} = 1$.

(8) (*) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is conditionally convergent. Let the sum be s. Let $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$. Now consider the rearranged series $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} - \frac{1}{7} \cdots$. Prove that this series converges to $\frac{1}{2}s$. (Hint: Look at the partial sum S_{3N} for this new series). The moral is that you cannot rearrange infinitely many terms in a conditionally convergent series and expect to get the same result.

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Tutorial Five.

Functions and their Properties.

(1) Calculate the following limits.
   (a) $\lim_{x\to 2} \frac{x}{x+3}$

   (b)$\lim_{x\to\infty}(\sqrt{x^2+4}-x)$

   (c) $\lim_{x\to 0} \frac{\sin(2x)}{x}$

   (d) $\lim_{x\to 2} \frac{x-2}{x^2-4}$

(2) Prove Theorem 2.2 in the lecture notes.

(3) Prove that the function $f(x) = x^2$ is continuous on any interval an that $\sin x$ is uniformly continuous on $\mathbb{R}$.

(4) Prove that every polynomial is continuous everywhere.

(5) A continuous function f is defined on an interval I and for every rational number $r \in I$, it satisfies $f(r) = r^2$. Prove that for all $x \in I$, $f(x) = x^2$

(6) Show that every polynomial of odd degree has at least one real root.

(7) Let f be a continuous function on an interval $[a, b]$, where $-\infty < a < b < \infty$. Suppose that for every $x \in I$ there exists a $y \in I$ such that $|f(y)| \leq \dfrac{1}{2}|f(x)|$. Prove that there exists an $\xi \in I$ such that $f(\xi) = 0$.

(8) Let $f : [a, b] \to [a, b]$ be continuous. Prove that f has a *fixed point*. That is, there exists $\xi \in [a, b]$ such that $f(\xi) = \xi$.

(9) Prove that if I is an interval and f is continuous on I, then $f(I) = \{y \in \mathbb{R} : f(x) = y, x \in I\}$ is also an interval. So continuous functions map intervals to intervals.

(10) Suppose that f is continuous on $\mathbb{R}$ and that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. Show that f attains its maximum and minimum values on $\mathbb{R}$.

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Tutorial Six.

The Derivative and its Applications.

(1) Calculate the derivative of $f(x) = \cos x$ from first principles. Then determine it a second way, using the fact that $\frac{d}{dx}\sin x = \cos x$. (Hint: Use a trig identity).

(2) Let $f(x) = \begin{cases} x, & x > 1 \\ x^2, & x \leq 1 \end{cases}$. Show that f is continuous everywhere, differentiable for $x \neq 1$, but not differentiable at $x = 1$.

(3) Let $f(x) = \begin{cases} 2x, & x \geq 1 \\ x^2 + 1, & x < 1 \end{cases}$. Show that f is differentiable at $x = 1$ and $f'(1) = 2$.

(4) Consider a polynomial P of degree n with the property that $P(\xi) = 0$ and $P'(\xi) = 0$. Prove that there is a polynomial Q of degree $n - 2$ such that $P(x) = (x - \xi)^2 Q(x)$.

(5) Use induction to prove that $\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$.

(6) Suppose that f is such that $\dfrac{d}{dx}(f(x^2)) = \dfrac{d}{dx}(f(x))^2$. Prove that $f'(1) = 0$ or $f(1) = 1$.

(7) Use the inverse function Theorem to give another proof of the fact that $\frac{d}{dx}e^x = e^x$.

(8) Prove that if $n > 1$, $f(x) = (x+1)^{1/n} - x^{1/n}$ decreases on $[0, \infty)$.

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Tutorial Seven.

Taylor Series.

(1) Prove that the Taylor series expansions about $a = 0$ for $\sin x$, $\cos x$ and e^x converge for all $x \in \mathbb{R}$.

(2) Obtain the Taylor series expansions of $f(x) = \sin x$ and $g(x) = \cos x$ about the point $a = \frac{\pi}{2}$. What do you notice about the powers in the expansion?

(3) Derive a Taylor series expansion for $f(x) = (1+x)^\alpha$, where α is not necessarily an integer. Prove that the series converges for $|x| < 1$.

(4) Use the series in the previous question to obtain an approximation to $\sqrt{3/2}$.

(5) Find a Taylor series expansion for $f(x) = \frac{x}{(1+x^2)^2}$ and determine its radius of convergence. Hint: There is an easy way to do this and a hard way.

(6) Find a Taylor expansion for $f(x) = \ln(1+x)$ and determine its radius of convergence.

(7) Determine the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{x^2}{n^2+1}$.

(8) (*) Let $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Prove that f is infinitely differentiable at 0 and $f^{(n)}(0) = 0$ for all n. Hence the Taylor series expansion of f around $a = 0$ equals f only at $x = 0$.

(9) Prove that the power series $\sum_{n=0}^{\infty} n^2 x^n$ is convergent for all $|x| < 1$ and calculate the sum. Hint compare with the geometric series.

(10) Suppose that the power series $y = \sum_{n=0}^{\infty} a_n x^n$ is convergent everywhere and satisfies $y' = y$. Prove that it must be the series for the function $y = a_0 e^x$.

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Tutorial Eight.

The Riemann Integral.

(1) Determine a formula for $\sum_{k=1}^{n} k^4$.

(2) Use a Riemann sum to compute the value of the integral

$$\int_0^1 (x^4 + 3x^2 + 2x)dx.$$

(3) Prove for any natural number n, that

$$\int_a^b x^n dx = \left[\frac{1}{n+1}x^{n+1}\right]_a^b.$$

(4) Use a Riemann sum to show that for any positive rational number α that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}}(1^\alpha + 2^\alpha + \cdots + n^\alpha) = \frac{1}{1+\alpha}.$$

(5) Suppose that f and g are continuous on $[a, b]$. Prove the Cauchy-Schwartz inequality

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx.$$

Hint: Consider the integral $\int_a^b (tf(x) + g(x))^2 dx$. Note that this is a quadratic in t and it is nonnegative. When is a quadratic nonnegative?

(6) Let g be continuous on the interval $[a, b]$ and suppose that $g(x) \geq 0$ for all $x \in [a, b]$. If $\int_a^b g(x)dx = 0$ prove that g is identically equal to zero on $[a, b]$.

(7) Suppose that f is twice differentiable on $[a, b$ and that f'' is continuous on $[a, b]$. Prove the formula

$$\int_a^b xf''(x)dx = bf'(b) - f(b) - (af'(a) - f(a)).$$

(8) (*) Let f be positive and continuous on $[1, \infty)$. Now suppose that

$$F(x) = \int_1^x f(t)dt \leq (f(x))^2, \ x \geq 1.$$

Prove that $f(x) \geq \frac{1}{2}(x-1)$ for $x \in [1, \infty)$. Hint: Consider the integral $\int_1^x \frac{F'(t)}{\sqrt{F(t)}}dt$.

(9) Prove that the improper Riemann integral $\int_0^\infty \frac{dx}{1+x^2}$ converges and determine its value.

(10) Prove that the improper Riemann integral $\int_0^1 \frac{dx}{\sqrt{x}}$ exists and determine its value.

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Tutorial Nine.

The Riemann Integral Continued.

(1) (*) Let f be positive, continuous and decreasing on $[1, \infty)$. Prove that the sequence

$$\Delta_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx$$

is decreasing and bounded below by zero and so converges.

(2) Use the previous question to prove that if $\int_1^\infty f(x)dx < \infty$, then the series $\sum_{n=1}^\infty f(n)$ is convergent. Conversely if the integral diverges, so does the infinite series.

(3) Prove that $\sum_{n=1}^\infty \frac{1}{n^\alpha}$ converges if $\alpha > 1$ and diverges otherwise.

(4) If f is continuous and increasing on $[0, \infty)$, prove that

$$\int_0^n f(x)dx \leq \sum_{k=1}^{n} f(k) \leq \int_1^{n+1} f(x)dx.$$

Show that $n \ln n - n \leq ln(n!) \leq (n+1)\ln(n+1) - n$. Conclude that $\frac{n^n}{n!} \leq e^n \leq \frac{(n+1)^{n+1}}{n!}$.

(5) Suppose that h is a positive continuous function on $[0, \infty)$. Let

$$H(x) = 1 + \int_0^x f(t)dt.$$

If $h(x) \geq H(x)$ show that for $x > 0$, $h(x) \geq e^x$.

Sequences of Functions.

(6) Prove that $f_n(x) = x + 1/n$ converges uniformly to $f(x) = x$ on $\mathbb{R}$. Prove that $f_n^2 \to f^2$ pointwise on $\mathbb{R}$, but the convergence is not uniform.

(7) Let $f_n(x) = x^{2n}(1 + x^{2n})^{-1}$. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f(x) = 0$ for $|x| < 1$, $f(1) = f(-1) = 1/2$. and $f(x) = 1$ for $|x| > 1$. So that each f_n is continuous, but f is not continuous at $x = 1$ and $x = -1$.

(8) Show that the series $\sum_{n=1}^\infty \frac{\cos(nx)}{n^4}$ is uniformly convergent on $[0, \infty)$.

(9) Prove that the series $\sum_{n=1}^\infty \frac{x^n}{n^2}$ is uniformly convergent on $[0, 1]$.

(10) Prove that $f_n(x) = nxe^{-nx^2} \to 0$ pointwise on $\mathbb{R}$, but the convergence is not uniform. Hint: Let $x = 1/\sqrt{n}$.
(11) Let f be continuously differentiable on $[-\pi, \pi]$. Show that $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(x)\sin(n\pi x)dx = 0$. Hint: Integrate by parts.
(12) Let $f_n(x) = \dfrac{x^2}{1+e^{-xn}}$. Determine $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)dx$.

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Tutorial Ten.

Review Problems.

(1) Let $x_0 = 1$ and define the sequence $x_{n+1} = x_n + \frac{1}{x_n}$. This is an example of a *continued fraction.* Prove that the sequence converges and the limit is the so called *golden ratio.* (Look up the golden ratio to see that you have the correct answer).

(2) Test the series $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$ for convergence using the integral test.

(3) If f and g are continuous on I, $a \in I$, prove that

$$\lim_{x \to a} (f \circ g)(x) = f(g(a)).$$

(4) Prove that a Lipschitz continuous function on an interval $[a, b]$ is uniformly continuous on $[a, b]$.

(5) If f is continuous and bounded on $\mathbb{R}$, prove that it is uniformly continuous on $\mathbb{R}$.

(6) Calculate the Taylor series expansion of $f(x) = \cos^2 x$.

(7) (*) Let $y'(x) = y(x)^2$ and suppose that $y(0) = 1$. By repeated applications of the chain rule, determine the first few terms of the Taylor series expansion of y.

(8) Consider the integral $\int_0^1 x^2 dx$. Find numbers A, B and C such that

$$\int_0^1 x^2 dx = Af(0) + Bf(1/2) + Cf(1)$$

where $f(x) = x^2$. What happens if you apply this rule to $f(x) = x^3$?

(9) The Gamma function is defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, x > 0$. Prove that $\Gamma(x + 1) = x\Gamma(x)$ and $\Gamma(n + 1) = n!$ for n a natural number.

(10) (i) Show that for $s > 1$,

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-nx} dx.$$

(ii) Prove that for $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$