

## Real Analysis 35007. Tutorial One.

### Manipulations with upper and lower bounds.

Recall that  $\inf(A)$  is the greatest lower bound of the set of real numbers  $A$  and  $\sup$  is the least upper bound of  $A$ .

- (1) Suppose that  $\xi > 0$  and  $S$  is a non empty set of real numbers bounded above. Prove that

$$\sup_{x \in S} \xi x = \xi \sup_{x \in S} x.$$

- (2) Suppose that  $S$  is non empty, bounded above and that  $S_0 \subseteq S$ . (So  $S_0$  is contained in  $S$ . It might equal  $S$ .) Prove that  $\sup S_0 \leq \sup S$ .

- (3) Suppose that  $S$  is non empty, bounded above and that  $\xi$  is any real number. Prove that  $\sup_{x \in S} (x + \xi) = \xi + \sup_{x \in S} x$ .

- (4) The distance between a point  $\xi$  and a set  $S$  is defined to be  $d(\xi, S) = \inf_{x \in S} |\xi - x|$ .

(a) If  $\xi \in S$  prove that  $d(\xi, S) = 0$ .

(b) If  $S$  is bounded above and  $\xi = \sup S$ , prove that  $d(\xi, S) = 0$ . If  $S$  is bounded below, and  $\xi = \inf S$  prove that  $d(\xi, S) = 0$ .

(c) If  $I$  is a closed interval, prove that  $d(\xi, I) = 0$  implies that  $\xi \in I$ . If  $I$  is open, prove that we can always find an  $\xi \notin I$  such that  $d(\xi, I) = 0$ .

### Limits.

Recall that a sequence  $\{a_n\}_{n=1}^{\infty}$  converges to a limit  $a$  as  $n \rightarrow \infty$  is for every  $\epsilon > 0$  we can find an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - a| < \epsilon$ .

Use the definition of a limit to establish the following.

(5)  $\lim_{n \rightarrow \infty} \frac{n}{2n+4} = \frac{1}{2}.$

(6)  $\lim_{n \rightarrow \infty} \frac{2n+1}{3n+2} = \frac{2}{3}.$

(7)  $\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0.$

Use properties of limits to show that

(8)  $\lim_{n \rightarrow \infty} \left( \frac{2n^3 - 3n}{5n^3 + 4n^2 - 2} \right) = \frac{2}{5}.$

(9)  $\lim_{n \rightarrow \infty} (\sqrt{n^2 + 4} - n) = 0.$

## Real Analysis 35007. Tutorial Two.

### Understanding Limits.

- (1) For what values of  $x$  does  $\lim_{n \rightarrow \infty} \frac{x + x^n}{1 + x^n}$  exist?
- (2) Suppose that  $\{y_n\}_{n=1}^{\infty}$  is a sequence of real numbers and  $y_n \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\{x_n\}_{n=1}^{\infty}$  be another sequence of real numbers. Suppose that for all  $n$ ,  $|x_n - l| \leq y_n$ . Prove that  $x_n \rightarrow l$ .
- (3) Prove that the sequence  $\left\{\left(1 + \frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$  converges. Note that if  $a_1, \dots, a_n$  are positive, then  $(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{k=1}^n a_k$ . (You will prove this later). Use your calculator to guess what the limit is.
- (4) (\*) Let  $x > 0$  and let  $N$  be the smallest natural number such that  $N > x$ . Prove that

$$\frac{x^n}{n!} \leq \frac{x^{N-1}}{(N-1)!} \left(\frac{x}{N}\right)^{n-N+1}, \quad n \geq N.$$

Conclude that  $x^n/n! \rightarrow 0$  as  $n \rightarrow \infty$ . This result is essential for proving the convergence of power series.

- (5) (\*) Let  $\alpha$  be any positive rational number and let  $|x| < 1$ . Show that there exists a natural number  $N$  such that

$$(1 + 1/N)^{\alpha+1} |x| \leq 1.$$

Deduce that

$$|n^{\alpha+1} x^n| \leq |N^{\alpha+1} x^N|,$$

for  $n \geq N$ . Hence show that  $n^{\alpha} x^n \rightarrow 0$  as  $n \rightarrow \infty$ . This is also important in establishing the convergence of certain kinds of series.

### Subsequences.

- (6) Find a convergent subsequence of  $\left\{\sin\left(\frac{\pi n}{2}\right)\right\}_{n=1}^{\infty}$ .
- (7) Suppose that  $\{x_n\}_{n=1}^{\infty}$  is a bounded sequence and for any  $N$ , we can find  $n \geq N$ , such that  $x_n \geq b$ . Show that  $x_n$  has a subsequence which converges to a limit  $l \geq b$ .
- (8) Find a convergent subsequence of  $\left\{\frac{3^n + (-2)^n}{3^n - 2^n}\right\}_{n=1}^{\infty}$ . What can you say about the limit in general?
- (9) It can be shown that  $n^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ . Suppose that we know that the limit exists, but we do not know its value. Determine the limit by considering the behaviour of the subsequence  $\{(2n)^{\frac{1}{2n}}\}$ .

### Real Analysis 35007. Tutorial Three.

#### lim sup and lim inf.

- (1) Consider the sequence  $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \dots$ . Determine the lim sup and lim inf for this sequence.
- (2) Let  $\{x_n\}_{n=1}^{\infty}$  be a bounded sequence with limit superior given by  $l$ . Let the limit inferior be  $L$ . Show that for any  $\epsilon > 0$  we can find  $N > 0$  such that for all  $n \geq N$ ,  $x_n < l + \epsilon$ . Formulate and prove the corresponding statement for  $x_n$  and  $L$ .

#### Cauchy Sequences.

- (3) Suppose that  $|x_{n+1} - x_n| \leq r^n$  where  $0 < r < 1$ . Prove that  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence.
- (4) We have a sequence defined by the recursive formula

$$x_{n+2} = (x_{n+1}x_n)^{1/2}.$$

Suppose that  $0 < a \leq x_1 \leq x_2 \leq b$ . Prove that  $a \leq x_n \leq b$  for all  $n \geq 0$ . Hence establish the inequality

$$|x_{n+2} - x_{n+1}| \leq \frac{b}{a+b} |x_{n+1} - x_n|.$$

Deduce that  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence and hence converges.

- (5) Let  $x_1 = a, x_2 = b$ . Set  $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$  for all  $n \geq 0$ . Prove that the sequence  $\{x_n\}_{n=1}^{\infty}$  converges.
- (6) How do calculators determine square roots? Most operations for determining function values are encoded into the hardware. Here is one algorithm. We let  $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ ,  $x_1 = x_0 > 0$ , for  $a > 0$ . Prove that the sequence is convergent and that its limit is  $\sqrt{a}$ . Use this to obtain an approximation to the square root of 2. Hint: Let  $y_n = \frac{x_n}{\sqrt{a}}$  and determine  $\frac{y_n - 1}{y_n + 1}$ . There are perfect squares involved.

#### The Bolzano-Weierstrass Theorem.

- (7) Show that every point in the interval  $[0, 1]$  is the limit of a subsequence of the sequence defined in Question one.
- (8) (\*) Given a set  $S$  of real numbers, let  $S_{\xi} = \{x; x \in S, x \neq \xi\}$ . We say that  $\xi$  is a limit point (or cluster point) of  $S$  if there is a sequence of points in  $S_{\xi}$  which converges to  $\xi$ . We can state a variation of the Bolzano-Weierstrass Theorem as follows. Every bounded set with an infinite number of elements contains at least one limit point. Prove this.

## Real Analysis 35007. Tutorial Four.

### Finite Sums.

- (1) Evaluate the sum  $\sum_{k=1}^N k$ . Prove your result by induction.
- (2) Find a formula for the sum  $\sum_{k=1}^N k^2$ . Prove your formula by induction.
- (3) Sums of the form  $\sum_{k=1}^N k^n$  can be shown to be given by polynomials in  $N$  of degree  $n+1$ . Use this to determine a formula for  $\sum_{k=1}^N k^3$ . You will need to solve a system of equations to find the coefficients of the fourth degree polynomial.

### Infinite Sums.

- (4) Prove that the series  $\sum_{n=1}^{\infty} \frac{1}{2n^2 + 3}$  is convergent.
- (5) Determine which of the following series converge and which diverge.
  - (i)  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ , (ii)  $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$ , (iii)  $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}$ ,
  - (iv)  $\sum_{n=1}^{\infty} n^{\alpha} x^n, |x| < 1, \alpha > 0$ , (v)  $\sum_{n=1}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}$ .
- (6) Prove that  $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)(n+5)} = \frac{23}{480}$ . Hint: Use partial fractions.
- (7) Prove that  $\sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)} = 1$ .
- (8) (\*\*) The series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  is conditionally convergent. Let the sum be  $s$ . Let  $S_N = \sum_{n=1}^N \frac{(-1)^{n+1}}{n}$ . Now consider the rearranged series  $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} - \frac{1}{7} \dots$ . Prove that this series converges to  $\frac{1}{2}s$ . (Hint: Look at the partial sum  $S_{3N}$  for this new series). The moral is that you cannot rearrange infinitely many terms in a conditionally convergent series and expect to get the same result.

## Real Analysis 35007. Tutorial Five.

### Continuous Functions and their Properties.

- (1) Calculate the following limits.
  - (a)  $\lim_{x \rightarrow 2} \frac{x}{x+3}$
  - (b)  $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 4} - x)$
  - (c)  $\lim_{x \rightarrow 0} \frac{\sin(2x)}{x}$
  - (d)  $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$
- (2) Prove Theorem 2.2 in the lecture notes.
- (3) Prove that the function  $f(x) = x^2$  is continuous on any interval and that  $\sin x$  is uniformly continuous on  $\mathbb{R}$ .
- (4) Prove that every polynomial is continuous everywhere.
- (5) A continuous function  $f$  is defined on an interval  $I$  and for every rational number  $r \in I$ , it satisfies  $f(r) = r^2$ . Prove that for all  $x \in I$ ,  $f(x) = x^2$ .
- (6) Show that every polynomial of odd degree has at least one real root.
- (7) Let  $f$  be a continuous function on an interval  $[a, b]$ , where  $-\infty < a < b < \infty$ . Suppose that for every  $x \in I$  there exists a  $y \in I$  such that  $|f(y)| \leq \frac{1}{2}|f(x)|$ . Prove that there exists an  $\xi \in I$  such that  $f(\xi) = 0$ .
- (8) (\*) Let  $f : [a, b] \rightarrow [a, b]$  be continuous. Prove that  $f$  has a *fixed point*. That is, there exists  $\xi \in [a, b]$  such that  $f(\xi) = \xi$ .
- (9) (\*) Prove that if  $I$  is an interval and  $f$  is continuous on  $I$ , then  $f(I) = \{y \in \mathbb{R} : f(x) = y, x \in I\}$  is also an interval. So continuous functions map intervals to intervals.
- (10) Suppose that  $f$  is continuous on  $\mathbb{R}$  and that  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$ . Show that  $f$  attains its maximum and minimum values on  $\mathbb{R}$ .

## Real Analysis 35007. Tutorial Six.

### The Derivative and its Properties.

- (1) Calculate the derivative of  $f(x) = \cos x$  from first principles. Then determine it a second way, using the fact that  $\frac{d}{dx} \sin x = \cos x$ . (Hint: Use a trig identity).
- (2) Let  $f(x) = \begin{cases} x, & x > 1 \\ x^2, & x \leq 1 \end{cases}$ . Show that  $f$  is continuous everywhere, differentiable for  $x \neq 1$ , but not differentiable at  $x = 1$ .
- (3) Let  $f(x) = \begin{cases} 2x, & x \geq 1 \\ x^2 + 1, & x < 1 \end{cases}$ . Show that  $f$  is differentiable at  $x = 1$  and  $f'(1) = 2$ .
- (4) Consider a polynomial  $P$  of degree  $n$  with the property that  $P(\xi) = 0$  and  $P'(\xi) = 0$ . Prove that there is a polynomial  $Q$  of degree  $n - 2$  such that  $P(x) = (x - \xi)^2 Q(x)$ .
- (5) Use induction to prove that  $\frac{d^n}{dx^n} (fg) = \sum_{k=0}^n \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$ .
- (6) Suppose that  $f$  is such that  $\frac{d}{dx}(f(x^2)) = \frac{d}{dx}(f(x))^2$ . Prove that  $f'(1) = 0$  or  $f'(1) = 1$ .
- (7) Use a result from the lectures to give another proof of the fact that  $\frac{d}{dx} e^x = e^x$ .
- (8) Let  $f(x) = x^x$ . Calculate  $f'(x)$ .
- (9) Let  $f(x) = \log_a x$ . Find  $f'(x)$ . Note that if  $a = e$ , then  $f'(x) = 1/x$ . What if  $a \neq e$ ?

## Real Analysis 35007. Tutorial Seven.

### Taylor Series.

- (1) Prove that the Taylor series expansions about  $a = 0$  for  $\sin x$ ,  $\cos x$  and  $e^x$  converge for all  $x \in \mathbb{R}$ .
- (2) Obtain the Taylor series expansions of  $f(x) = \sin x$  and  $g(x) = \cos x$  about the point  $a = \frac{\pi}{2}$ . What do you notice about the powers in the expansion?
- (3) Derive a Taylor series expansion for  $f(x) = (1+x)^\alpha$ , where  $\alpha$  is not necessarily an integer. Prove that the series converges for  $|x| < 1$ .
- (4) Use the series in the previous question to obtain an approximation to  $\sqrt{3/2}$ .
- (5) Find a Taylor series expansion for  $f(x) = \frac{x}{(1+x^2)^2}$  and determine its radius of convergence. Hint: There is an easy way to do this and a hard way.
- (6) Find a Taylor expansion for  $f(x) = \ln(1+x)$  and determine its radius of convergence.
- (7) Determine the interval of convergence for the series  $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$  and  $\sum_{n=1}^{\infty} \frac{x^2}{n^2+1}$ .
- (8) (\*) Let  $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ . Prove that  $f$  is infinitely differentiable at 0 and  $f^{(n)}(0) = 0$  for all  $n$ . Hence the Taylor series expansion of  $f$  around  $a = 0$  equals  $f$  only at  $x = 0$ .
- (9) Prove that the power series  $\sum_{n=0}^{\infty} n^2 x^n$  is convergent for all  $|x| < 1$  and calculate the sum. Hint compare with the geometric series.
- (10) Suppose that the power series  $y = \sum_{n=0}^{\infty} a_n x^n$  is convergent everywhere and satisfies  $y' = y$ . Prove that it must be the series for the function  $y = a_0 e^x$ .

**Real Analysis 35007. Tutorial Eight.**

**Limits and the Mean Value Theorem.**

- (1) Use L'Hôpital's rule to calculate the following limits.
  - (i)  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ .
  - (ii)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$
  - (iii)  $\lim_{z \rightarrow -1} \frac{z + 1}{z^5 + 1}$ .
  - (iv)  $\lim_{x \rightarrow 0} \frac{\tan x}{x}$ .
  - (v)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{4x^2 - \pi^2}$ .
- (2) Prove that  $|\ln z - \ln w| \leq 3|z - w|$  for  $\frac{1}{3} \leq z < w \leq 3$ .
- (3) Prove that  $|\sin x - \sin y| \leq |x - y|$  using the Mean Value Theorem.
- (4) Suppose we know that  $f(x)$  is continuous and differentiable on the interval  $[-7, 0]$ , that  $f(-7) = 3$  and that  $f'(x) \leq 2$ . What is the largest possible value for  $f(0)$ ?
- (5) Let  $f$  be continuous on  $[1, 3]$  and differentiable on  $(1, 3)$ . Suppose that, for all  $x \in (1, 3)$ ,  $1 \leq f'(x) \leq 2$ . Prove that  $2 \leq f(3) - f(1) \leq 4$ .
- (6) The velocity of a train is kept between 40 and 50 kilometers per hour. during a trip of 200 kilometers. What can you say about the duration of the trip?
- (7) Let  $f$  be a polynomial. Suppose that  $f$  has a double root at  $a$  and at  $b$ . Show that  $f'(x)$  has at least three roots in  $[a, b]$ .
- (8) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable for all  $x \in \mathbb{R}$ . Use the Mean Value Theorem to show that if  $f'(x) < 0$  for all  $x \in \mathbb{R}$  then  $f$  is strictly decreasing.
- (9) Let  $f$  be twice differentiable on  $(a, b)$  and suppose  $f$  vanishes at three distinct points in  $(a, b)$ . Prove that there is a point  $x_0 \in (a, b)$  at which  $f''(x_0) = 0$ .
- (10) Let  $y = \sum_{n=0}^{\infty} a_n x^n$ . Find the values of  $a_n$  if the function satisfies the differential equation  $y' = xy$ ,  $y(0) = 1$ .



**Real Analysis 35007. Tutorial Nine.**

**The Riemann Integral.**

- (1) Determine a formula for  $\sum_{k=1}^n k^4$ .
- (2) Use a Riemann sum to compute the value of the integral

$$\int_0^1 (x^4 + 3x^2 + 2x)dx.$$

- (3) Prove for any natural number  $n$ , that

$$\int_a^b x^n dx = \left[ \frac{1}{n+1} x^{n+1} \right]_a^b.$$

- (4) Use a Riemann sum to show that for any positive rational number  $\alpha$  that

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\alpha+1}} (1^\alpha + 2^\alpha + \cdots + n^\alpha) = \frac{1}{1+\alpha}.$$

- (5) Suppose that  $f$  and  $g$  are continuous on  $[a, b]$ . Prove the Cauchy-Schwartz inequality

$$\left( \int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx.$$

Hint: Consider the integral  $\int_a^b (tf(x) + g(x))^2 dx$ . Note that this is a quadratic in  $t$  and it is nonnegative. When is a quadratic nonnegative?

- (6) Let  $g$  be continuous on the interval  $[a, b]$  and suppose that  $g(x) \geq 0$  for all  $x \in [a, b]$ . If  $\int_a^b g(x)dx = 0$  prove that  $g$  is identically equal to zero on  $[a, b]$ .

- (7) Suppose that  $f$  is twice differentiable on  $[a, b]$  and that  $f''$  is continuous on  $[a, b]$ . Prove the formula

$$\int_a^b x f''(x) dx = b f'(b) - f(b) - (a f'(a) - f(a)).$$

- (8) (\*) Let  $f$  be positive and continuous on  $[1, \infty)$ . Now suppose that

$$F(x) = \int_1^x f(t)dt \leq (f(x))^2, \quad x \geq 1.$$

Prove that  $f(x) \geq \frac{1}{2}(x-1)$  for  $x \in [1, \infty)$ . Hint: Consider the integral  $\int_1^x \frac{F'(t)}{\sqrt{F(t)}} dt$ .

- (9) Prove that the improper Riemann integral

$$\int_0^\infty \frac{dx}{(1+x^2)(4+x^2)}$$

converges and determine its value.

- (10) Prove that the improper Riemann integral  $\int_0^1 \frac{dx}{\sqrt{x}}$  exists and determine its value.

# Real Analysis 35007. Tutorial Ten.

## The Riemann Integral Continued.

- (1) (\*) Let  $f$  be positive, continuous and decreasing on  $[1, \infty)$ . Prove that the sequence

$$\Delta_n = \sum_{k=1}^n f(k) - \int_1^n f(x)dx$$

is decreasing and bounded below by zero and so converges.

- (2) Use the previous question to prove that if  $\int_1^\infty f(x)dx < \infty$ , then the series  $\sum_{n=1}^\infty f(n)$  is convergent. Conversely if the integral diverges, so does the infinite series.

- (3) Prove that  $\sum_{n=1}^\infty \frac{1}{n^\alpha}$  converges if  $\alpha > 1$  and diverges otherwise.

- (4) If  $f$  is continuous and increasing on  $[0, \infty)$ , prove that

$$\int_0^n f(x)dx \leq \sum_{k=1}^n f(k) \leq \int_1^{n+1} f(x)dx.$$

Show that  $n \ln n - n \leq \ln(n!) \leq (n+1) \ln(n+1) - n$ . Conclude that  $\frac{n^n}{n!} \leq e^n \leq \frac{(n+1)^{n+1}}{n!}$ .

- (5) Suppose that  $h$  is a positive continuous function on  $[0, \infty)$ . Let

$$H(x) = 1 + \int_0^x h(t)dt.$$

If  $h(x) \geq H(x)$  show that for  $x > 0$ ,  $h(x) \geq e^x$ .

## Sequences of Functions.

- (6) Prove that  $f_n(x) = x + 1/n$  converges uniformly to  $f(x) = x$  on  $\mathbb{R}$ . Prove that  $f_n^2 \rightarrow f^2$  pointwise on  $\mathbb{R}$ , but the convergence is not uniform.
- (7) Let  $f_n(x) = x^{2n}(1 + x^{2n})^{-1}$ . Let  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . Show that  $f(x) = 0$  for  $|x| < 1$ ,  $f(1) = f(-1) = 1/2$ . and  $f(x) = 1$  for  $|x| > 1$ . So that each  $f_n$  is continuous, but  $f$  is not continuous at  $x = 1$  and  $x = -1$ .
- (8) Show that the series  $\sum_{n=1}^\infty \frac{\cos(nx)}{n^4}$  is uniformly convergent on  $[0, \infty)$ .
- (9) Prove that the series  $\sum_{n=1}^\infty \frac{x^n}{n^2}$  is uniformly convergent on  $[0, 1]$ .

- (10) Let  $f$  be continuously differentiable on  $[-\pi, \pi]$ . Show that  $\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin(n\pi x) dx = 0$ . Hint: Integrate by parts.
- (11) Let  $f_n(x) = \frac{x^2}{1 + e^{-xn}}$ . Determine  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ .

## Real Analysis 35007. Tutorial Eleven

### Evaluating Integrals

- (1) Evaluate the integral  $\int_0^\infty e^{-xy} \sin(ax) dx$ . Use integration by parts.
- (2) Calculate the value of the integral  $\int_0^\infty e^{-\lambda x^3} dx$  in terms of the Gamma function.
- (3) Evaluate the integral  $\int_0^1 t^{-2/3} (1-t)^{-1/3} dt$ .
- (4) Use the substitution  $x = \frac{\pi}{2} - y$  to evaluate the integral

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^m x} dx, \quad m > 0.$$

- (5) Evaluate the integral  $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$  by using the substitution  $x = \pi - y$ .
- (6) Determine the value of  $\int_0^1 \frac{\ln(1+x)}{x^2+1} dx$  by using the substitution  $x = \frac{1-t}{1+t}$ .
- (7) Use differentiation under the integral sign to calculate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

This is called a Frullani integral.

- (8) Use differentiation under the integral sign to evaluate the integral

$$\int_0^\infty e^{-\alpha x^2 - \frac{\beta}{x^2}} dx, \quad \alpha, \beta > 0.$$

Put  $y = \sqrt{\alpha x}$  first.

- (9) Calculate  $\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx$  by differentiating the integral

$$\int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(\alpha \tan x)}{\tan x} dx.$$

- (10) Find the value of  $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx$  from  $\int_0^\infty \frac{\ln(1+\alpha x^2)}{1+x^2} dx$ .

- (11) Evaluate the integral  $\int_0^1 \frac{\ln x}{(x+1)^2} dx$  by expanding the denominator in a power series and integrating term by term.

- (12) Determine a recurrence formula for the integral

$$I_n = \int_0^1 \frac{\ln x}{(x+1)^n} dx, \quad n > 2.$$

Note that

$$I_n = \int_0^1 \frac{(1+x-x)\ln x}{(x+1)^n} dx$$

and use integration by parts on the integral  $\int_0^1 \frac{x \ln x}{(x+1)^n} dx$ . What is the value of  $I_3$ ?

- (13) Use an infinite series to evaluate the integral  $\int_0^1 \ln x \ln(x+1) dx$ .

- (14) Use an infinite series to determine the value of

$$\int_0^\infty \frac{x^2}{\cosh^2(x^2)} dx.$$

You can evaluate the series using Wolfram Alpha or Mathematica.

- (15) Show that the double integral  $\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy$  exists and use a geometric series to calculate it. Hence determine the value of  $\int_0^1 \frac{\ln(1-x)}{x} dx$ . Now evaluate  $\int_0^1 \int_0^1 \frac{(xy)^a}{1-xy} dx dy$  by the same means. Here  $a \geq 0$ . These are examples of Beuker's integrals and they arise in number theory. In particular one can ask how well an irrational number can be approximated by a rational number and in this problem Beuker's integrals appear.

### Real Analysis 35007. Review and Challenge Problems

- (1) Let  $x_0 = 1$  and define the sequence  $x_1 = 1 + 1$ ,  $x_2 = 1 + \frac{1}{1+1}$ ,  
 $x_3 = 1 + \frac{1}{1+\frac{1}{1+1}}$ ,

$$x_4 = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+1}}}$$

etc. This is an example of a *continued fraction*. Prove that the sequence converges and the limit is the so called *golden ratio*. (Look up the golden ratio to see that you have the correct answer).

- (2) Test the series  $\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$  for convergence using the integral test.

- (3) If  $f$  and  $g$  are continuous on  $I$ ,  $a \in I$ , prove that

$$\lim_{x \rightarrow a} (f \circ g)(x) = f(g(a)).$$

- (4) Prove that a Lipschitz continuous function on an interval  $[a, b]$  is uniformly continuous on  $[a, b]$ .

- (5) Calculate the Taylor series expansion of  $f(x) = \cos^2 x$ .

- (6) (\*) Let  $y'(x) = y(x)^2$  and suppose that  $y(0) = 1$ . By repeated applications of the chain rule, determine the first few terms of the Taylor series expansion of  $y$ .

- (7) Consider the integral  $\int_0^1 f(x)dx$ . Find numbers  $A, B$  and  $C$  such that

$$\int_0^1 f(x)dx = Af(0) + Bf(1/2) + Cf(1)$$

where  $f(x) = 1, x, x^2$  in turn. You should have a system of three linear equations to solve. This is Simpson's rule. What happens if you apply this rule to  $f(x) = x^3$ ?

- (8) Establish the result

$$\int \frac{1}{a + b \sin(2x)} dx = \frac{1}{\sqrt{a^2 - b^2}} \int \frac{dy}{y^2 + 1} = \frac{1}{\sqrt{a^2 - b^2}} \tan^{-1} y.$$

from the lecture notes.

- (9) The Gamma function is defined by  $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ ,  $x > 0$ . Prove that  $\Gamma(x+1) = x\Gamma(x)$  and  $\Gamma(n+1) = n!$  for  $n$  a natural number.

(10) (i) Show that for  $s > 1$ ,

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-nx} dx.$$

(ii) Prove that for  $s > 1$ ,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

(11) Prove that

$$\int_1^\infty \frac{\ln x}{x^2 + 1} dx = G,$$

where  $G$  is Catalan's constant.

(12) Show that

$$\int_0^\pi \ln(t + \alpha \cos x) dx = \pi \ln \left( \frac{t + \sqrt{1 - \alpha^2}}{2} \right), \quad |\alpha| < 1.$$

You will need

$$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left( \frac{(a - b) \tan \left( \frac{x}{2} \right)}{\sqrt{a^2 - b^2}} \right), \quad a^2 > b^2.$$

(13) Calculate the integral

$$f(a) = \int_{-\infty}^{\infty} \frac{\sin^2(ax)}{x^2} dx, \quad a > 0.$$

You will need to differentiate under the integral sign, use a trigonometric identity and then use a simple substitution to turn  $f'(a)$  into the Dirichlet integral.