Tutorial Six Solutions

(1) $f(x) = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \cos x \underbrace{\cancel{\frac{(\cos h - 1)}{h}}}_{0} - \sin x \lim_{h \to 0} \underbrace{\cancel{\frac{\sin h}{h}}}_{1}$$

$$= -\sin x$$

Or $\quad \cos x = \sin\left(x + \frac{\pi}{2}\right)$

So $\dfrac{d}{dx} \cos x = \dfrac{d}{dx} \sin\left(x + \dfrac{\pi}{2}\right) = \cos\left(x + \dfrac{\pi}{2}\right)$

$$= \cos x \cos \frac{\pi}{2} - \sin x \sin \frac{\pi}{2}$$

$$= -\sin x.$$

(2) x and x^2 are continuous

Now $\displaystyle\lim_{x \to 1^-} f(x) = 1 = \lim_{x \to 1^+} f(x)$

So f is continuous at $x = 1$.

x and x^2 are differentiable everywhere.

Now

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{1+h-1}{h} = 1.$$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^-} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0^-} 2 + h = 2 \neq 1.$$

So $\displaystyle\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ does not exist.

(3) $$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2(1+h) - 2}{h} = 2$$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 + 1 - (1+1)}{h} = \lim_{h \to 0^-} \frac{2h + h^2}{h} = 2$$

(4) $P(x) = \sum_{k=0}^{n} a_k x^k$

Now $P(\xi)=0$. So ξ is a root. We factorise $P(x)$ to get $P(x) = (x-\xi)P_{n-1}(x)$

Now $P'(x) = P_{n-1}(x) + (x-\xi)P'_{n-1}(x)$

Since

$$P'(\xi) = P_{n-1}(\xi) + 0 = 0$$

It follows that ξ is a root of $P_{n-1}(x)$. Hence $P_{n-1}(x) = (x-\xi)Q(x)$ where Q is a polynomial of degree $n-2$. Thus

$$P(x) = (x-\xi)^2 Q(x)$$

(5) $(fg)' = f'g + fg'$. ($n=1$ case)

Now suppose $\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$

Then $\frac{d^{n+1}}{dx^{n+1}}(fg) = \frac{d}{dx}\left(\frac{d^n}{dx^n}(fg)\right) = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dx}\left(\frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} g^{(n-k+1)}(x) f^{(k)}(x) + \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(x) g^{(n-k)}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} g^{(n-k+1)}(x) f^{(k)}(x) + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)}(x) g^{(n+1-k)}(x)$$

$$= \binom{n}{0} g^{(n-k+1)}(x) f(x) + \binom{n}{n} f^{(n+1)}(x) g(x)$$

$$+ \sum_{k=1}^{n} \binom{n}{k} g^{(n-k+1)}(x) f^{(k)}(x) + \sum_{k=1}^{n} \binom{n}{k-1} f^{(k)}(x) g^{(n+1-k)}(x)$$

$$= \binom{n+1}{0} g^{(n-k+1)}(x) f(x) + \binom{n+1}{n+1} f^{(n+1)}(x) g(x)$$

$$+ \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] g^{(n+1-k)}(x) f^{(k)}(x)$$

Now $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. This is not hard to prove. Try it

$$\text{So } (fg)^{(n+1)} = \binom{n+1}{0} g^{(n-k+1)}(x) f(x) + \binom{n+1}{n+1} f^{(n+1)}(x) g(x)$$

$$+ \sum_{k=1}^{n} \binom{n+1}{k} g^{(n+1-k)}(x) f^{(k)}(x)$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)}(x) g^{(n+1-k)}$$

Now this is true for $n=1$. So it is true for all n by induction.

6) $\frac{d}{dx}(f(x^2)) = 2x f'(x^2)$ by the chain rule

$\frac{d}{dx}(f(x))^2 = 2f(x) f'(x)$

Since $2xf'(x^2) = 2f(x) f'(x)$

$x f'(x^2) = f(x) f'(x)$

Take $x = 1$ Then $f'(1) = f(1) f'(1)$

$\Rightarrow f(1) = 1$ or $f'(1) = 0$

(7) e^x is the inverse of $\ln x$

$\ln x = \int_1^x \frac{dy}{y}$

$\ln(e^x) = x = \int_1^x \frac{dy}{y}$

$$\frac{d}{dx}\ln(e^x) = \frac{1}{e^x}\frac{d}{dx}(e^x) = \frac{d(x)}{dx} = 1$$

$$\therefore \frac{d}{dx}(e^x) = e^x$$

Q8 $f(x) = x^x$. We observe that $x = e^{\ln x}$.
So $f(x) = e^{x\ln x}$

$$\therefore \quad f'(x) = \frac{d}{dx} e^{x\ln x} = \frac{d}{du} e^u \frac{du}{dx}, \quad u = x\ln x$$

$$= e^u\left(\ln x + x.\frac{1}{x}\right) = (1+\ln x)\, e^{x\ln x}$$

$$= (1+\ln x)\, x^x.$$

Q9 $f(x) = \log_a x$. Now $a^{\log_a x} = x$ by definition of the logarithm.
We can also write

$$a = e^{\ln a}.$$

So $x = e^{(\ln a)\log_a x}$.
Thus

$$\frac{dx}{dx} = 1 = \frac{d}{dx} e^{(\ln a)\log_a x}$$

$$= \ln a \frac{d(\log_a x)}{dx} e^{(\ln a)\log_a x}$$

$$\therefore \quad 1 = \ln a \frac{d}{dx}(\log_a x)\, x$$

or $\quad \dfrac{d}{dx}(\log_a x) = \dfrac{1}{(\ln a)\, x}.$