Real Analysis Tutorial 3 Solutions.

Q1. The sequence is $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, ...$ Notice that there is a lot of repetition in the sequence. In fact every rational number in $(0,1)$ occurs infinitely often. The terms are of the form n/m, where $m \geq 2$ is a positive integer and $1 \leq n \leq m-1$.

It is clear that the sequence is bounded below by zero and bounded above by 1. So the lim inf of the sequence cannot be smaller than 0 and the lim sup cannot be larger than 1. An obvious subsequence is $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ which converges to zero. Another subsequence is $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...$ which converges to 1. So if we denote the sequence of x_n we have

$$\liminf x_n = 0,$$
$$\limsup x_n = 1.$$

Q2.

We do this by contradiction. Suppose the statement is false. Then for some $\epsilon > 0$ for each N we can find $n > N$ such that $x_n \geq l + \epsilon$. The Bolzano-Weierstrass Theorem says that every bounded sequence has a convergent subsequence. Since we can find a subsequence between L and $\lambda + \epsilon$ we can find a subsequence with limit $\bar{l} \geq \lambda + \epsilon$. But this contradicts the definition of l.

For L, we can say that for any $\epsilon > 0$ we can find an N such that for any $n > N$ we have $x_n > L - \epsilon$. The proof is basically the same. Just replace l with L and reverse the inequality signs in the first proof.

The purpose of the next few questions is to introduce you to the idea of iteration as a solution method. This idea is quite simple, but very powerful. It is one of the most widely used techniques in analysis.

Q3. We have a sequence x_n such that $|x_{n+1} - x_n| \leq r^n$, where $0 \leq r \leq 1$. Clearly

$$\lim_{n\to\infty} |x_{n+1} - x_n| \leq \lim_{n\to\infty} r^n = 0.$$

However this does not mean that x_n is a Cauchy sequence as the Harmonic series example shows. However we can use our favourite adding and subtracting trick. Let $n > m$. Then

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leq r^{n-1} + r^{n-2} + \cdots + r^m \\
&= r^m(1 + r + \cdots + r^{n-m-1}).
\end{aligned}$$

This is a geometric progression and we have

$$\begin{aligned} r^m(1 + r + \cdots + r^{n-m-1}) &= r^m \frac{1 - r^{m-n}}{1-r} \\ &= \frac{r^m - r^n}{1-r} \to 0, \end{aligned}$$

as $n, m \to \infty$. So the sequence is Cauchy. To make this a little more precise, (though this is good enough), let $\epsilon > 0$. Note that $0 < r < 1$ so $r^m > r^n$ if $n > m$. Thus

$$\begin{aligned} \left| \frac{r^m - r^n}{1-r} \right| &\leq \frac{1}{1-r}(r^m + r^n) \\ &\leq \frac{2r^m}{1-r} < \epsilon, \end{aligned}$$

provided that $r^m < \frac{1-r}{2}\epsilon$. Choose N such that $m > N$ implies

$$r^m < \frac{1-r}{2}\epsilon.$$

Since $r^m \to 0$, this is obviously possible. Thus $n, m \geq N$ implies $|x_n - x_m| < \epsilon$ so x_n is a Cauchy sequence.

Q4. The sequence is defined by $x_{n+2} = (x_n x_{n+1})^{1/2}$. First note that if $x_1, x_2 \leq b$ then $x_3 \leq (b^2)^{1/2} = b$. Then $x_4 \leq (b^2)^{1/2} = b$ etc. So that $x_n \leq b$ for all n. The proof that $x_n \geq a$ is basically the same, but with the inequality reversed.

Now to prove that we have a Cauchy sequence, we make it easier by squaring both sides. Hence

$$x_{n+2}^2 = x_n x_{n+1}.$$

Subtract x_{n+1}^2 from both sides. Thus

$$\begin{aligned} x_{n+1}^2 - x_{n+1}^2 &= x_n x_{n+1} - x_{n+1}^2 \\ &= x_{n+1}(x_n - x_{n+1}). \end{aligned}$$

This implies that

$$x_{n+2} - x_{n+1} = \frac{x_{n+1}(x_n - x_{n+1})}{x_{n+2} + x_{n+1}}$$

since $x_{n+1}^2 - x_{n+1}^2 = (x_{n+2} - x_{n+1})(x_{n+2} + x_{n+1})$. Thus

$$|x_{n+2} - x_{n+1}| = \frac{|x_{n+1}|}{|x_{n+2} + x_{n+1}|} |x_n - x_{n+1}|.$$

Now $a \leq x_n \leq b$. So

$$\frac{|x_{n+1}|}{|x_{n+2} + x_{n+1}|} \leq \frac{b}{a+b}.$$

To see why this is, notice that $|x_{n+1}| \leq b$, so the largest value the numerator can take is b. Hence the smallest value the denominator can take is $a+b$, since the smallest value x_{n+2} can take is a. Hence we have the inequality

$$|x_{n+2} - x_{n+1}| \leq \frac{b}{a+b}|x_{n+1} - x_n|.$$

Or

$$|x_{n+1} - x_n| \leq \frac{b}{a+b}|x_n - x_{n-1}|.$$

Thus

$$|x_n - x_{n-1}| \leq \left(\frac{b}{a+b}\right)|x_{n-1} - x_{n-2}|.$$

Hence

$$|x_{n+1} - x_n| \leq \left(\frac{b}{a+b}\right)^2 |x_{n-1} - x_{n-2}|.$$

Continuing we have

$$\begin{aligned}|x_{n+1} - x_n| &\leq \left(\frac{b}{a+b}\right)^{n-1} |x_2 - x_1| \\ &= r^n \frac{|x_2 - x_1|}{r},\end{aligned}$$

where $r = \frac{b}{a+b} < 1$. Define the sequence $\tilde{x}_n = \frac{r}{|x_2 - x_1|}x_n$. Then we have

$$|\tilde{x}_{n+1} - \tilde{x}_n| \leq r^n,$$

so that $\tilde{x}_n$ is Cauchy by the previous question and hence convergent. Since multiplying a convergent sequence by a constant produces another convergent sequence, x_n is convergent.

Q5.

Here we have $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$. We assume $a \neq b$. (What happens if $a = b$?). This is similar to the previous two questions. The simple observation is that

$$\begin{aligned}x_{n+2} - x_{n+1} &= \frac{1}{2}(x_{n+1} + x_n) - x_{n+1} \\ &= \frac{1}{2}(x_n - x_{n+1}).\end{aligned}$$

Thus

$$\begin{aligned}|x_{n+2} - x_{n+1}| &= \frac{1}{2}|x_{n+1} - x_n| = \left(\frac{1}{2}\right)^2 |x_n - x_{n-1}| \cdots \\ &= \frac{1}{2^n}|x_2 - x_1| = \frac{|b-a|}{2^n}.\end{aligned}$$

Now let $\tilde{x}_n = \frac{1}{|b-a|}x_n$ and then apply the result of Q3.

Q6

First we assume the limit exists. Suppose $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$. Then we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}(x_n + \frac{a}{x_n}).$$

We have $a > 0$ and $x_0 > 0$. Thus $x = \frac{1}{2}(x + \frac{a}{x})$ or $x^2 = a$. Since x must be positive, $x = \sqrt{a}$. Now we make the substitution $y_n = \dfrac{x_n}{\sqrt{a}}$. We will show that $\lim_{n\to\infty} y_n = 1$.

A little bit of algebra gives

$$y_{n+1} - 1 = \frac{(y_n - 1)^2}{2y_n},$$
$$y_{n+1} + 1 = \frac{(y_n + 1)^2}{2y_n}.$$

(Just substitute y_n into the sequence and rearrange the above expressions). So that

$$\frac{y_{n+1} - 1}{y_{n+1} + 1} = \frac{(y_n - 1)^2}{(y_n + 1)^2}.$$

Consequently

$$\frac{y_n - 1}{y_n + 1} = \frac{(y_{n-1} - 1)^2}{(y_{n-1} + 1)^2}.$$

Which yields

$$\frac{y_{n+1} - 1}{y_{n+1} + 1} = \frac{(y_n - 1)^2}{(y_n + 1)^2} = \left(\frac{y_{n-1} - 1}{y_{n-1} + 1}\right)^4 \cdots$$
$$= \left(\frac{y_0 - 1}{y_0 + 1}\right)^{2n}.$$

Now if $|y_0 - 1| < |y_0 + 1|$, which it will be with our conditions, we have

$$\left(\frac{y_0 - 1}{y_0 + 1}\right)^{2n} \to 0,$$

as $n \to \infty$. Thus $\lim_{n\to\infty} \frac{y_{n+1}-1}{y_{n+1}+1} = 0$, so $y_n \to 1$.

Now if we take $a = 2$, $x_0 = 1$ we get $x_2 = \frac{3}{2}, x_2 = 17/12, x_3 = 577/408$, etc. In fact $x_4 = 1.41421$ and the first five decimal places are correct. This is a very efficient method for computing square roots.

Q7.

It should be clear that every rational number in $[0, 1]$ appears in the sequence x_n in Q1 infinitely often. Now we can approximate any real number as a sequence of rationals. In fact we can define the real numbers this way. To illustrate

$$\frac{\sqrt{2}}{2} = 0.707106...$$

which can be approached by the sequence

$$7/10, 70/100, 707/1000, 7071/10000...$$

which is a subsequence of our sequence.

To make this more precise, let $x \in (0,1)$. We can find a term x_{n_1} of the sequence x_n such that $x-1 < x_{n_1} < x+1$. (Try this with $x = \frac{\sqrt{2}}{2}$ to convince yourself that you can do this). Then we find an $n_2 > n_1$ such that $x - \frac{1}{2} < x_{n_2} < x + \frac{1}{2}$, an $n_3 > n_2$ such that $x - \frac{1}{3} < x_{n_3} < x + \frac{1}{3}$, etc. Clearly the subsequence $x_{n_k} \to x$.

Q8

Let $\{x_n\}$ be a subsequence of S. Since $\{x_n\}$ is bounded it contains a convergent subsequence by the Bolzano-Weierstrass Theorem. Let this subsequence be x_{n_k} and suppose that $\lim_{k\to\infty} x_{n_k} = \xi \in S$. We can think of x_{n_k} as a sequence in its own right and so by definition ξ is a limit point of S.