Tutorial 11 RA solutions

1) $$I = \int_0^\infty e^{-xy}\sin(ax)\,dx = -\frac{1}{y}e^{-xy}\overset{0}{\cancel{\sin(ax)}}\Big]_0^\infty + \frac{a}{y}\int_0^\infty e^{-xy}\cos(ax)\,dx$$

$$= \left[-\frac{a}{y^2}e^{-xy}\cos(ax)\right]_0^\infty - \frac{a^2}{y^2}\int_0^\infty e^{-xy}\sin(ax)\,dx$$

$$= \frac{a}{y^2} - \frac{a^2}{y^2}I$$

$$\therefore \left(1+\frac{a^2}{y^2}\right)I = \frac{a}{y^2} \quad \text{So } I = \frac{a}{y^2}\,\frac{y^2}{a^2+y^2}$$

$$= \frac{a}{a^2+y^2}$$

(2) $\int_0^\infty e^{-\lambda x^3}dx$ Put $\lambda x^3 = t$.

$$x = (t/\lambda)^{1/3} \qquad \left(\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt\right)$$

$$dx = \frac{1}{3\lambda^{1/3}}t^{-2/3}dt$$

$$\int_0^\infty e^{-\lambda x^3}dx = \int_0^\infty \frac{1}{3\lambda^{1/3}}t^{-\frac{2}{3}}e^{-t}dt$$

$$= \frac{1}{3\lambda^{1/3}}\Gamma\left(\frac{1}{3}\right)$$

3) $$\int_0^1 t^{-2/3}(1-t)^{-1/3}dt = B\left(\frac{1}{3},\frac{2}{3}\right)$$

$$= \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3}+\frac{2}{3})} = \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$$

$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$

$x-1 = -\frac{2}{3}$

$x = \frac{1}{3}$, $y-1 = -1/3$

$y = 2/3$

Now $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$

So $\Gamma(\frac{1}{3})\Gamma(1-\frac{1}{3}) = \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}}$.

$$\therefore \int_0^1 t^{-\frac{2}{3}}(1-t)^{-\frac{1}{3}}dt = \frac{2\pi}{\sqrt{3}}.$$

(4) $I=\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^m x}$, $m>0$. Put $x=\frac{\pi}{2}-y$

$$I=-\int_{\frac{\pi}{2}}^{0} \frac{dy}{1+\tan^m(\frac{\pi}{2}-y)} = \int_0^{\pi/2} \frac{\sin^m y}{\cos^m y+\sin^m y}\,dy$$

Now $$I=\int_0^{\pi/2} \frac{\cos^m x}{\sin^m x+\cos^m x}\,dx$$

$$\therefore I+I=2I=\int_0^{\pi/2} \frac{\cos^m y+\sin^m y}{\cos^m y+\sin^m y}\,dy$$

$$=\int_0^{\pi/2} dy$$

$$=\frac{\pi}{2}$$

$\therefore I=\pi/4$ for all $m>0$.

(5)
$I=\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$. $x=\pi-y$

$$-\int_{\pi}^{0} \frac{(\pi-y)\sin(\pi-y)}{1+\cos^2(\pi-y)}\,dy$$

$$=\int_0^{\pi} \frac{(\pi-y)\sin y}{1+\cos^2 y}\,dy$$

$\therefore 2I=\int_0^{\pi} \frac{\pi\sin y}{1+\cos^2 y}\,dy$ $\cos y=u$
$du=-\sin y\,dy$

$$2I=\int_{-1}^{1} \frac{\pi}{1+u^2}\,du$$

$$=\pi\left[\tan^{-1}u\right]_{-1}^{1} = \frac{\pi^2}{2}.$$

$$\therefore I=\frac{\pi^2}{4}.$$

(6) $I=\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$. $x=\frac{1-t}{1+t}$.

$$dx=-\frac{2}{(1+t)^2}\,dt$$

$$x^2+1=\left(\frac{1-t}{1+t}\right)^2+1$$

$$\frac{1}{1+x^2} = \frac{(1+t)^2}{(1-t)^2+(1+t)^2} = \frac{(1+t)^2}{2(1+t^2)}$$

Now $\ln(1+x) = \ln\left(1+\frac{1-t}{1+t}\right) = \ln\left(\frac{1+t+1-t}{1+t}\right)$

$$= \ln\left(\frac{2}{1+t}\right)$$

$$I = \int_1^0 \ln\left(\frac{2}{1+t}\right)\frac{(1+t)^2}{2(1+t^2)}\,\frac{-2}{(1+t)^2}\,dt$$

$$= \int_0^1 \frac{\ln 2 - \ln(1+t)}{1+t^2}\,dt$$

$$\therefore\ 2I = \int_0^1 \frac{\ln 2}{1+t^2}\,dt = \ln 2\left[\tan^{-1} t\right]_0^1$$

$$= \frac{\pi \ln 2}{4}$$

So $\underline{I} = \frac{\pi}{8}\ln 2.$

(7) $F(a,b) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$

$$\frac{\partial F}{\partial a} = -\int_0^\infty e^{-ax}\,dx$$

$$= \lim_{R\to\infty} \left.\frac{e^{-ax}}{a}\right|_0^R = \frac{-1}{a}.$$

$\therefore F = -\ln a + c(b)$ c is some constant depending on b.

$$\frac{\partial F}{\partial b} = \int_0^\infty e^{-bx}\,dx = -\lim_{R\to\infty}\left[\frac{e^{-bx}}{b}\right]_0^\infty = \frac{1}{b}.$$

So $F(a,b) = \ln b + \bar{c}(a)$

Comparing $F(a,b) = \ln\left(\frac{b}{a}\right)$

(8) $I = \int_0^\infty e^{-\alpha x^2 - \frac{\beta}{x^2}}\,dx$, $\alpha, \beta > 0$. $y = \sqrt{\alpha}\,x$, $x = \frac{y}{\sqrt{\alpha}}$

$\alpha x^2 = y^2$ $dx = \frac{1}{\sqrt{\alpha}}\,dy$

$$-\frac{\beta}{x^2} = -\frac{\beta\alpha}{y^2}.$$

So $\int_0^\infty e^{-\alpha x^2-\frac{\beta}{x^2}}dx = \frac{1}{\sqrt{\alpha}}\int_0^\infty e^{-y^2-\frac{p^2}{y^2}}dy \quad p=\sqrt{\alpha\beta}$

Let $J(p)=\int_0^\infty e^{-y^2-\frac{p^2}{y^2}}dy.$

Then $J'(p) = -2\int_0^\infty p\, e^{-y^2-\frac{p^2}{y^2}}\frac{dy}{y^2}.$

Now put $z=\frac{p}{y}$. $dz=-\frac{p}{y^2}dy$

$\therefore \frac{p^2}{y^2}=z^2,\ \frac{1}{z^2}=\frac{y^2}{p^2}.$ So $\frac{p^2}{z^2}=y^2$

Hence $J'(p)=-2\int_0^\infty e^{-z^2-\frac{p^2}{z^2}}\frac{dz}{z^2}$

$=-2J(p).$

This is a separable ODE.

$J(p)=Ae^{-2p}$

$J(0)=\int_0^\infty e^{-y^2}dy=\frac{\sqrt{\pi}}{2}.$

Thus $\int_0^\infty e^{-\alpha x^2-\beta/x^2}dx=\frac{\sqrt{\pi}}{2\sqrt{\alpha}}e^{-2\sqrt{\alpha\beta}}.$

(9) $F(\alpha)=\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\alpha\tan x)}{\tan x}dx$

$F'(\alpha)=\int_0^{\pi/2}\frac{dx}{1+\alpha^2\tan^2 x}$

Now put $u=\alpha\tan x$ $\quad du=\alpha\sec^2 x\,dx$

we know $1+\tan^2 x=\sec^2 x$

So $du=\alpha(1+\tan^2 x)dx$

$\therefore dx=\frac{du}{\alpha+\frac{1}{\alpha}u^2}=\frac{\alpha\,du}{u^2+\alpha^2}$

$\therefore F'(\alpha)=\int_0^\infty\frac{\alpha\,du}{(\alpha^2+u^2)(1+u^2)}$

$$\frac{\alpha}{(u^2+\alpha^2)(1+u^3)} = \frac{\alpha}{(\alpha^2-1)(u^2+1)} - \frac{\alpha}{(\alpha^2-1)(u^2+\alpha^2)}$$

$$= \frac{\alpha}{\alpha^2-1}\left[\frac{1}{u^2+1} - \frac{1}{u^2+\alpha^2}\right]$$

So $$\int_0^\infty \frac{\alpha\,du}{(u^2+1)(u^2+\alpha^2)} = \frac{\alpha}{\alpha^2-1}\left[\tan^{-1}u - \frac{1}{\alpha}\tan^{-1}\left(\frac{u}{\alpha}\right)\right]_0^\infty$$

$$= \frac{\alpha}{\alpha^2-1}\left[\frac{\pi}{2} - \frac{\pi}{2\alpha}\right]$$

$$= \frac{\pi}{2}\,\frac{\alpha}{\alpha^2-1}\left(\frac{\alpha-1}{\alpha}\right) = \frac{\pi}{2}\,\frac{1}{\alpha+1}$$

$$\therefore \quad F(\alpha) = \frac{\pi}{2}\int \frac{d\alpha}{\alpha+1} = \frac{\pi}{2}\ln(1+\alpha) + C$$

Now $$F(0) = \int_0^{\pi/2} \frac{\tan^{-1}0}{\tan x}\,dx = 0$$

So $$\frac{\pi}{2}\ln 1 + C = 0 \quad \therefore \quad C = 1$$

Thus $$F(\alpha) = \frac{\pi}{2}\ln(1+\alpha)$$

$$F(1) = \int_0^{\pi/2} \frac{x}{\tan x}\,dx = \frac{\pi}{2}\ln 2.$$

10)

$$F(\alpha) = \int_0^\infty \frac{\ln(1+\alpha x^2)}{1+x^2}\,dx \qquad F(0) = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

$$\therefore \quad F'(\alpha) = \int \frac{x^2}{(1+x^2)(1+ax^2)}\,dx$$

$$\int_0^\infty \frac{x^2\,dx}{(1+x^2)(1+ax^2)} = \int_0^\infty \left(\frac{1}{(\alpha-1)(1+x^2)} + \frac{1}{(1-\alpha)(1+ax^2)}\right)dx$$

$$= \left[\frac{\tan^{-1}x}{\alpha-1}\right]_0^\infty + \frac{1}{1-\alpha}\left.\frac{\tan^{-1}(\sqrt{\alpha}\,x)}{\sqrt{\alpha}}\right]_0^\infty$$

$$= \frac{\pi}{2}\left(\frac{1}{\alpha-1}\left(1-\frac{1}{\sqrt{\alpha}}\right)\right) = \frac{\pi}{2}\,\frac{\sqrt{\alpha}-1}{\sqrt{\alpha}(\alpha-1)}$$

$$= \frac{\pi}{2}\cdot\frac{\sqrt{\alpha}-1}{\sqrt{a}(\sqrt{\alpha}-1)(\sqrt{\alpha}+1)} = \frac{\pi}{2(\alpha+\sqrt{\alpha})}$$

Put $\alpha = u^2$

$$\frac{\pi}{2}\int \frac{d\alpha}{\alpha+\sqrt{\alpha}} = \frac{\pi}{2}\int \frac{2u\,du}{u^2+u} = \pi\int \frac{du}{u+1}$$

$$= \pi\ln(1+u) + C$$

So $F(\alpha) = \pi\ln(1+\sqrt{\alpha}) + C$

Now $F(0) = \frac{\pi}{2} = \pi\ln 1 + C$

$C = \frac{\pi}{2}$.

Thus $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = F(1) = \pi\ln 2 - \frac{\pi}{2}$.

(ii) $\int_0^1 \frac{\ln x}{(1+x)^2}\,dx = I$.

Now $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ $\quad |x|<1$

So $\frac{1}{(1+x)^2} = -\frac{d}{dx}\frac{1}{(1+x)} = -\frac{d}{dx}\left(1 - x + x^2 - x^3 + x^4 - \cdots\right)$

$= 1 - 2x + 3x^2 - 4x^3 + \cdots$

for $|x|<1$

Now $\int_0^1 x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x\Big|_0^1 - \int_0^1 \frac{x^n}{n+1}\,dx$

$\left(\lim_{x\to 0} x^n\ln x = 0\right)$ $\quad = -\frac{x^{n+1}}{(n+1)^2}\Big|_0^1 = -\frac{1}{(n+1)^2}$

So $\int_0^1 \frac{\ln x}{(1+x)^2}\,dx = \int_0^1 \ln x\left(\sum_{k=0}^{\infty}(-1)^k(k+1)x^k\right)dx$

$$= \sum_{k=0}^{\infty}(-1)^k(k+1)x^k\int_0^1 x^k\ln x\,dx$$

$$= \sum_{k=0}^{\infty}(-1)^{k+1}\frac{(k+1)}{(k+1)^2}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k+1}$$

$$= -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right) = -\ln 2$$

(12) $I_n = \int_0^1 \frac{\ln x}{(x+1)^n} \, dx$

So $$I_n = \int_0^1 \frac{(1+x-x)\ln x}{(x+1)^n} \, dx$$

$$= \int_0^1 \frac{\ln x \, dx}{(x+1)^{n-1}} - \int_0^1 \frac{x \ln x}{(1+x)^n} \, dx$$

$$= I_{n-1} - \int_0^1 \frac{x \ln x}{(1+x)^n} \, dx$$

Now $$\int_0^1 \frac{x \ln x \, dx}{(1+x)^n} = \left. \frac{x \ln x}{(1-n)(1+x)^{n-1}} \right]_0^1 - \int_0^1 \frac{(\ln x + 1)}{(1-n)(1+x)^{n-1}} \, dx$$

$$= -\int_0^1 \frac{\ln x}{(1-n)(1+x)^{n-1}} + \int_0^1 \frac{dx}{(1-n)(1+x)^{n-1}}$$

$$= \frac{-1}{1-n} I_{n-1} + \left[\frac{1}{(1-n)n(1+x)^n} \right]_0^1$$

$$= \frac{-1}{1-n} I_{n-1} + \frac{1}{n(1-n)} \left[\frac{1}{2^n} - 1 \right]$$

$$\therefore \; I_n = I_{n-1} - \left(\frac{-1}{1-n} I_{n-1} \right) - \frac{1}{n(1-n)} \left(\frac{1}{2^n} - 1 \right)$$

$$= \frac{(1-n)+1}{1-n} I_{n-1} - \frac{1}{n(1-n)} \left(\frac{1}{2^n} - 1 \right)$$

$$= \frac{n}{n-1} I_{n-1} + \frac{1}{n(n-1)} \left(\frac{1}{2^n} - 1 \right)$$

$$I_3 = \frac{3 I_2}{2} + \frac{1}{3.2} \left(\frac{1}{8} - 1 \right)$$

$$= \frac{3}{2} \left(-\ln 2 \right) + \frac{1}{6} \left(\frac{1}{8} - 1 \right)$$

13) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$$\therefore \int_0^1 \ln x \ln(1+x)\,dx = \sum_{n=1}^{\infty}(-1)^{n+1}\int_0^1 (\ln x)\, x^n\,dx$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{-1}{(n+1)^2}$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{(n+1)^2}$$

$$= \frac{\pi^2}{12} - 1$$

(14) $I = \int_0^{\infty}\frac{x^2}{\cosh^2(x^2)}\,dx$. First put $u = x^2$

$du = 2x\,dx$. $dx = \frac{du}{2\sqrt{u}}$

So $I = \frac{1}{2}\int_0^{\infty}\frac{u^{1/2}}{(\cosh u)^2}\,du$

Now $\cosh u = \frac{1}{2}(e^u + e^{-u}) = \frac{1}{2}e^u(1+e^{-2u})$

So $\cosh^2 u = \frac{1}{4}e^{2u}(1+e^{-2u})^2$

Hence $I = \frac{1}{2}\int_0^{\infty}\frac{u^{1/2}}{\frac{1}{4}e^{2u}(1+e^{-2u})^2}\,du$

Now $\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$

Giving $I = \frac{4}{2}\int_0^{\infty} u^{1/2}e^{-2u}(1 - 2e^{-2u} + 3e^{-4u} - 4e^{-6u} +-)\,du$

$$= 2\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^{\infty} n u^{1/2} e^{-2nu}\,du$$

Now $\int_0^{\infty} n u^{1/2} e^{-2nu}\,du = \frac{1}{4}\sqrt{\frac{\pi}{2}}\cdot\frac{1}{n}$.

So $I = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\sqrt{\frac{\pi}{2}}\,\frac{1}{n} = \sqrt{\frac{\pi}{2}}\,\frac{\ln 2}{2}$.

(15) $\int_0^1\int_0^1 \frac{dxdy}{1-xy} = \int_0^1\int_0^1 \left(1+xy+(xy)^2+(xy)^3+- \right)dxdy$

we evaluate first $= \int_0^1 \left(x + \frac{x^2}{2}y + \frac{x^3}{3}y^2 + \frac{x^4}{4}y^3 +- \right)\Big]_0^1 dy$

$= \int_0^1 \left(1 + \frac{y}{2} + \frac{y^2}{3} + \frac{y^3}{4} +- \right)dy$

$= \left[y + \frac{y^2}{4} + \frac{y^3}{9} + \frac{y^4}{16} + - - \right]_0^1$

$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \sim \quad = \frac{\pi^2}{6} < \infty$

Reverse order of integration and result is the same.

Now $\int_0^1\int_0^1 \left(\frac{dy}{1-xy}\right) dx = -\int_0^1 \frac{\ln(1-x)}{x} dx$. (*)

So $\int_0^1 \frac{\ln(1-x)}{x} dx = -\pi^2/6$

We evaluated a similar integral to (*) before. We also have

$$\int_0^1\int_0^1 \frac{dx}{1-xy} dy = -\int_0^1 \frac{\ln(1-y)}{y} dy$$

Now $\int_0^1\int_0^1 \frac{(xy)^a}{(1-xy)} dxdy = \int_0^1 \left[(xy)^a + (xy)^{a+1} + - - . \right] dxdy$

$= \int_0^1 \left[\frac{x^{a+1}}{a+1} y^a + \frac{x^{a+2}y^{a+1}}{a+2} + - - + \right]_0^1 dy$

$= \int_0^1 \left(\frac{y^a}{a+1} + \frac{y^{a+1}}{(a+2)} + \frac{y^{a+2}}{(a+3)} + - \right) dy$

$= \sum_{n=1}^{\infty} \frac{1}{(a+n)^2}$