Tutorial 10 Solutions

(1) Since f decreases
$$f(k+1) \le \int_k^{k+1} f(x)\,dx \le f(k)$$
Thus
$$\Delta_{n+1} - \Delta_n = \left\{ \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x)\,dx \right\} - \left\{ \sum_{k=1}^{n} f(k) - \int_1^{n} f(x)\,dx \right\}$$
$$= f(n+1) - \int_n^{n+1} f(x)\,dx \le f(n+1) - f(n+1) = 0$$
So $\{\Delta_n\}$ decreases. Also
$$\Delta_n = \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx$$
$$\ge \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} f(k) = f(n) > 0$$
So Δ_n is bounded below

(2) From (1) if f is positive, continuous and decreasing on $[1,\infty)$ then either both
$$\sum_{n=1}^{\infty} f(n) \text{ and } \int_1^{\infty} f(x)\,dx$$
converge or they both diverge.

This is the <u>integral</u> test and it's very useful

(3) $\int_1^\infty \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p} \Big|_1^\infty = \frac{1}{1-p} < \infty$ for $p>1$.

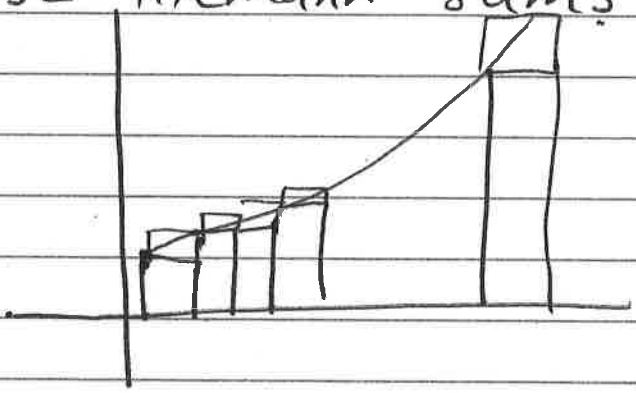
So if $p>1$ $\sum_{n=1}^\infty \frac{1}{n^p} < \infty$, by integral test

If $p=1$ $\int_1^\infty \frac{dx}{x} = \ln x \Big|_1^\infty = \infty$

So $\sum_{n=1}^\infty \frac{1}{n} = \infty$

If $p<1$ $\int_1^\infty \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p} \Big|_1^\infty = \infty$ $\therefore$ $\sum_{n=1}^\infty \frac{1}{n^p} = \infty$ for $p<1$

(4) Use Riemann sums



Partition $[0,n]$ into $[0,1), [1,2), \ldots [n-1,n]$ f increases, so f is smallest at the left end point. $x_0=0$, $x_1=1$, x_2 --- $x_n = n-1$.

It is largest at right end points.

So $\int_0^n f(x)dx \le \sum_{k=1}^n f(k)$.

Now use the partition $[1,2), [2,3), \ldots [n,n+1]$

Clearly, if we take the Riemann lower sum

$$\int_1^{n+1} f(x)dx \ge \sum_{k=1}^n f(k)$$

(Note $x_i - x_{i-1} = 1$ for all these sums.)

Thus $\int_0^n f(x)dx \le \sum_{k=1}^n f(k) \le \int_1^{n+1} f(x)dx$.

$\ln n! = \ln 1 + \ln 2 + \cdots + \ln n.$

Take $f(x) = \ln x$

$$\int_0^n \ln x \, dx = x\ln x - x \Big]_0^n = n\ln n - n$$

Note $\lim_{x \to 0} x \ln x = 0$

So by the inequality

$$n\ln n - n \leq \sum_{k=1}^{n} \ln k = \ln 1 + \ln 2 + \cdots + \ln n$$
$$= \ln(n!)$$

Similarly $\ln n! \leq \int_1^{n+1} \ln x \, dx = (n+1)\ln(n+1) - n$

So $\ln n^n - n \leq \ln(n!) \leq \ln(n+1)^{n+1} - n$

Now take the exponential of both sides

$$e^{\ln n^n - n} \leq e^{\ln(n!)} \leq e^{\ln(n+1)^{n+1} - n}$$

or $\dfrac{n^n}{e^n} \leq n! \leq \dfrac{(n+1)^{n+1}}{e^n}$

Divide by $n!$ and multiply by e^n.

$$\frac{n^n}{n!} \leq e^n \leq \frac{(n+1)^{n+1}}{n!}$$

(5) $H(x) = 1 + \int_0^x h(t)\,dt$

$$H'(x) = h(x) \geq h(x)$$

Thus $\displaystyle\int_0^x \frac{H'(t)}{H(t)}\,dt \geq \int_0^x dt = x.$ $(x > 0)$

But $\displaystyle\int_0^x \frac{H'(t)}{H(t)}\,dt = \ln H(x) \geq x.$ But $H(0) = 1$

$\therefore H(x) \geq e^x$ $h(x) \geq H(x) \geq e^x$

(6) $f_n(x) = x + \frac{1}{n} \to x$ as $n \to \infty$

Note if $f(x) = x$

$$|f_n(x) - f(x)| = |x + \tfrac{1}{n} - x| = \tfrac{1}{n}$$

So if $\cancel{n > \frac{1}{\varepsilon}}$ $n > N \geq \frac{1}{\varepsilon}$, $(N \in \mathbb{N})$

then $|f_n(x) - f(x)| < \varepsilon$ all x.

$\therefore$ $f_n \to f$ uniformly

Now $f_n^2 = (x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$.

Thus $|f_n^2(x) - f^2(x)| = |x^2 + \frac{2x}{n} + \frac{1}{n^2} - x^2|$

$$= \left|\frac{2x}{n} + \frac{1}{n^2}\right|$$

Let $\varepsilon > 0$ and for simplicity let $x > 0$.

Then $\left|\frac{2x}{n} + \frac{1}{n^2}\right| = \frac{2x}{n} + \frac{1}{n^2} < \varepsilon$

$$\Rightarrow \quad 2xn + 1 < \varepsilon n^2$$

$$\varepsilon n^2 - 2xn - 1 > 0$$

$$\Rightarrow \quad n > \frac{2x + \sqrt{4x^2 - 4\varepsilon}}{2\varepsilon}.$$

Thus n depends on x, So convergence is not uniform

7) $f_n(x) = \dfrac{x^{2n}}{1 + x^{2n}}$. If $|x| < 1$ $x^{2n} \to 0$ by previous work

So for $|x| < 1$ $f_n \to 0$

$f_n(1) = \dfrac{1}{1+1} = \dfrac{1}{2}$, $f_n(-1) = \dfrac{(-1)^{2n}}{1 + (-1)^{2n}} = \dfrac{1}{2}$.

For $|x| > 1$ $\lim\limits_{n \to \infty} \dfrac{x^{2n}}{1 + x^{2n}} = \lim\limits_{n \to \infty} \dfrac{1 + x^{2n} - 1}{1 + x^{2n}}$

$= 1 - \lim\limits_{n \to \infty} \dfrac{1}{1 + x^{2n}} = 1$, as $\dfrac{1}{x^{2n}} \to 0$ as $n \to \infty$ $|x| > 1$.

Therefore each f_n is continuous, but f is not.

(8) $|f_n(x)| = \left|\frac{\cos(nx)}{n^4}\right| \leq \frac{1}{n^4}$

and $\sum_{n=1}^{\infty} \frac{1}{n^4} < \infty$. So by M test

$\sum f_n$ is uniformly convergent

(9) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. $|f_n(x)| = \left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$, $x \in [0,1]$

and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$.

So $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

(10) $f_n(x) = nxe^{-nx^2}$, $f_n(0) = 0 \to 0$

$\frac{nx}{e^{nx^2}} \to 0$ as $n \to \infty$ by L'Hopital

Let $f = 0$

$|f_n(x) - f(0)| = |nxe^{-nx^2}|$

$\leq |nx| < \varepsilon$

So n depends on x, and convergence is not uniform.

11) $b_n = \int_{-\pi}^{\pi} f(x) \sin(n\pi x)\,dx$

$= -\frac{f(x)\cos(n\pi x)}{n\pi}\Big]_{-\pi}^{\pi}$

$+ \int_{-\pi}^{\pi} f'(x) \frac{\cos(n\pi x)}{n\pi}\,dx$

So $|b_n| \leq \frac{1}{n\pi} 2|f(x)| + \int_{\pi}^{\pi} \frac{|f'(x)|}{n\pi}\,dx$

Now f, f' are bounded on $[-\pi, \pi]$ by continuity, so

$$|b_n| \leq \frac{\operatorname{Max}|f(x)|}{n\pi} + \frac{\operatorname{Max}|f'(x)|}{n\pi}\int_{-\pi}^{\pi} dx$$

$\rightarrow 0$ as $n \rightarrow \infty$

(12) $f_n(x) = \dfrac{x^2}{1+e^{-xn}} \leq x^2$ which is integrable.

$$\text{So } \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n$$
$$= \int_0^1 x^2\,dx = \frac{1}{3}$$

For the proof of this result, you need to do Lebesgue integration and Fourier analysis.