Final Tute Solutions

(1) $x = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}$ $\therefore$ $x = 1 + \frac{1}{x}$

or $x^2 - x - 1 = 0$

$x > 0$. So $x = \frac{1+\sqrt{5}}{2}$

This number is called the Golden Ratio
It is the hardest rational number to approximate with a fraction

$x_0 = 1$, $x_1 = x_0 + \frac{1}{x_0}$, $x_2 = 1 + \frac{1}{1+1}$

$x_3 = 1 + \cfrac{1}{1+\cfrac{1}{1+1}}$ etc. See last page for proof of convergence

2) $f(x) = \frac{x^2}{x^4+1}$. $\int_1^\infty \frac{x^2}{x^4+1}\,dx < \infty$.

To see this note that $\frac{x^2}{x^4+1} = \frac{1}{x^2+\frac{1}{x^2}} \le \frac{1}{x^2}$ So

$\int_1^\infty \frac{dx}{x^2+\frac{1}{x^2}} \le \int_1^\infty \frac{dx}{x^2} < \infty$. By integral test

$$\sum_{n=1}^{\infty} \frac{n^2}{n^4+1} < \infty.$$

3) f, g are continuous. So $f \circ g$ is continuous
By definition of continuity $\lim_{x \to a} f(g(x)) = f(g(a))$

4) $|f(x) - f(y)| \le K|x-y|$ on $[a,b]$.
Let $\varepsilon > 0$. $\delta = \frac{\varepsilon}{K}$

$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{K} K = \varepsilon$

δ is independent of x, y. So f is uniformly continuous.

(5) $f(x) = \cos^2 x$.

a) $\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$

$= 1 - x^2 + \left(\frac{1}{4!} + \frac{1}{4!} + \left(\frac{1}{2!}\right)^2\right)x^4 \dots$

or

$f(0) = 1$, $f'(x) = -2\cos x \sin x = -\sin(2x)$

So $f'(0) = 0$

$f''(x) = -2\cos(2x) \therefore f''(0) = -2$

$f'''(x) = 4\sin(2x) \quad f'''(0) = 0$

$f^{iv}(x) = 8\cos 2x \quad f^{(iv)}(0) = 8$ etc

So

$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots$

$= 1 - x^2 + \frac{8}{6!}x^4 \dots$

(6) $y' = y(x)^2 \qquad y'(0) = (1)^2 = 1$.

$y'' = 2yy' = 2y(x)^3 \quad \therefore y''(0) = 2$

$y''' = 6y^2y' = 6y^4 \quad \therefore y'''(0) = 6$

$\therefore y(x) = 1 + x + x^2 + 6\frac{x^3}{3!} + \dots$

$= 1 + x + x^2 + x^3 + \dots$

$= \frac{1}{1-x}$

(7) $\int_0^1 x^2\,dx = \frac{1}{3} = Af(0) + Bf(\tfrac{1}{2}) + Cf(1)$

$= \frac{B}{4} + C$

$\int_0^1 x\,dx = \frac{1}{2} = A + \frac{B}{2} + C$

$\int_0^1 1\,dx = 1 = A + B + C$.

Get $A = \frac{1}{6}$, $B = \frac{2}{3}$, $C = \frac{1}{6}$

This is simpsons rule

$\int_0^1 x^3 dx = \frac{1}{4}$. Now if $f(x) = x^3$

$$\frac{1}{6}\left(f(0) + 4f\left(\tfrac{1}{2}\right) + f(1)\right) = \frac{1}{4}.$$

So Simpson's rule gives the exact rule.

8) $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, $x > 0$

$\Gamma(1) = 1$. This is easy

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -e^{-t} t^x \Big]_0^\infty + x\int_0^\infty t^{x-1} e^{-t} dt$$

$$= 0 + x\Gamma(x).$$

$$\therefore \quad \Gamma(x+1) = x\Gamma(x)$$

$\therefore$ $\Gamma(2) = 1.\Gamma(1) = 1$; $\Gamma(3) = 2\Gamma(2) = 1 \times 2$

$\Gamma(4) = 3\Gamma(3) = 1 \times 2 \times 3$

$\Gamma(5) = 4\Gamma(4) = 1 \times 2 \times 3 \times 4$ etc

$\Gamma(n+1) = n!$

9)

$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$. Put $t = nx$

$$\Gamma(s) = \int_0^\infty (nx)^{s-1} e^{-nx} n dx$$

$$= n^s \int_0^\infty x^{s-1} e^{-nx} dx$$

or

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-nx} dx$$

11) $$\sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-nx} dx$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \sum_{n=1}^\infty e^{-nx} dx$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty x^s \frac{e^{-x}}{1-e^{-x}} dx$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x-1} dx.$$

) Define the sequence $x_{n+1} = 1 + \frac{1}{x_n}$, $x_1 = 1$.
Let $b_n = x_n b_{n-1}$, $b_0 = 1$.

$\therefore$ $x_n = \frac{b_n}{b_{n-1}}$.

So $b_{n+1} = x_{n+1} b_n = b_n\left(1 + \frac{1}{x_n}\right) = b_n\left(1 + \frac{b_{n-1}}{b_n}\right)$

or $b_{n+1} = b_n + b_{n-1}$. This is the same as $b_{n+1} - b_n - b_{n-1} = 0$. Let $b_n = \lambda^n$

Then

$$\lambda^{n+1} - \lambda^n - \lambda^{n-1} = 0 \Rightarrow \lambda^2 - \lambda - 1 = 0.$$

Thus $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Hence $b_n = A\lambda_1^n + B\lambda_2^n$, some A, B.

Now $|\lambda_2| = .618...$

So $\lambda_2^n \rightarrow 0$ as $n \rightarrow \infty$

Thus $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{A\lambda_1^n + B\lambda_2^n}{A\lambda_1^{n-1} + B\lambda_2^{n-1}}$$

$$= \lambda_1 = \frac{1+\sqrt{5}}{2}.$$ This is the golden ratio.

Continued fractions often reduce to the solution of a quadratic. There is a whole theory of such things.