Tutorial 4 Solutions

1) We let $\sum_{k=1}^{n} k = an^2 + bn$

Then $n=1$ gives $a+b=1$.

$n=2$ gives $1+2=3 = a2^2+2b$

$$= 4a+2b=3$$

$$a+b=1.$$

$b=-a+1$. Hence $4a+2(1-a) = 2+2a = 3$

$\therefore\ 2a=1$, $a=\frac{1}{2}$, $b=1/2$

or

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1).$$

Now assume this is true for $n=N$.

Then

$$\sum_{k=1}^{N+1} k = \sum_{k=1}^{N} k + N+1$$

$$= \frac{1}{2}N(N+1) + N+1 = \frac{(N+1)}{2}(1+N+1)$$

$$= \frac{(N+1)}{2}(N+2)$$

Thus if result holds for $n=N$, it holds for $n=N+1$. It holds for $N=1$, so it holds for all n by induction.

2) Let $\sum_{k=1}^{n} k^2 = ak^3+bk^2+ck$

$$\left.\begin{aligned} n=1 &\Rightarrow 1 = a+b+c. \\ n=2 &\Rightarrow 1+4 = 8a+4b+2c \\ n=3 &\Rightarrow 1+4+9 = 27a+9b+3c \end{aligned}\right\} a=\frac{1}{3},\ b=\frac{1}{2},\ c=\frac{1}{6}$$

Giving

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$= \frac{n}{6}(2n+1)(n+1)$$

Now suppose that it is true for $n = N$.
Then
$$\sum_{k=1}^{N} k^2 + (N+1)^2 = \frac{N}{6}(2N+1)(N+1) + (N+1)^2$$
$$= (N+1)\left[N+1 + \frac{N(2N+1)}{6}\right]$$
$$= \frac{(N+1)}{6}\left[6N+6+N(2N+1)\right]$$
$$= \frac{N+1}{6}(N+2)(2N+3)$$

So again, result is true for $n = N+1$ as long as it is true for $n = N$. However it is true for $n=1$, so it is true for all n.

3) $\sum_{k=1}^{n} k^3 = ak^4 + bk^3 + ck^2 + dk$ algebra gives
$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2$$
or the famous result
$$(1+2+ \dots + n)^2 = 1^3 + 2^3 + \dots n^3.$$

4) $\sum_{n=1}^{\infty} \frac{1}{2n^2+3}$ converges. Use the comparison test.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. We have $2n^2+3 > n^2$
So $\frac{1}{2n^2+3} < \frac{1}{n^2}$.
Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we see that
$$\sum_{n=1}^{\infty} \frac{1}{2n^2+3} < \infty.$$

(5) (i) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ Use ratio test

Let $a_n = \frac{(n!)^2}{(2n)!}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^2}{(2n+2)!} \Big/ \frac{(n!)^2}{(2n)!}$

$$= \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \frac{(n+1)^2 \cancel{(n!)^2} \cdot \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(n!)^2}\cancel{(2n)!}}$$

$$= \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n^2+2n+1}{4n^2+4n+2}$$

$$= \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4+4/n+2/n^2} \to \frac{1}{4} < 1$$

So series converges.

(ii) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$. $a_n = \frac{(n!)^2 x^n}{(2n)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{(2n+2)(2n+1)} x^2 \to \frac{1}{4} x^2$$

So we have convergence for $|x^2| < 4$ or $x < 2$. For $|x| > 2$ series diverges.

(iii) Notice $\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{(\sqrt{n+1}-\sqrt{n})}{n} \frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})}$

$$= \frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges if $\alpha > 1$.

$$n(\sqrt{n+1}+\sqrt{n}) > n^{1+1/2}$$

$$\therefore \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n^{1+1/2}}$$

So $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ converges by comparison test.

(iv) $\sum_{n=1}^{\infty} n^{\alpha} x^{n}$ $\qquad a_n = n^{\alpha} x^{n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{\alpha} x^{n+1}}{n^{\alpha} x^{n}}\right|$$

$$= \left(\frac{n+1}{n}\right)^{\alpha} |x|.$$

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{\alpha}|x| = |x| \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{\alpha}$$

$$= |x| \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\alpha}$$

$$= |x| < 1.$$ So series converges.

Note we have divergence for $|x| \geq 1$.

(v). $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ $\qquad a_n = \frac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!}\right| = \frac{|x|}{n+1}$$

So $\lim_{n\to\infty} \frac{|x|}{n+1} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0$

all x So converges for all x

In fact $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

(5) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+3)(n+5)} = \sum_{n=1}^{\infty}\left[\frac{1}{8}\left(\frac{1}{n+5}+\frac{1}{n+1}-\frac{2}{n+3}\right)\right]$

$$= \frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{n+5}+\frac{1}{n+1}-\frac{2}{n+3}\right)$$

$$= \frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{n+5}-\frac{1}{n+3}+\frac{1}{n+1}-\frac{1}{n+3}\right]$$

$$= \frac{1}{8}\left(\sum_{n=1}^{\infty}\left[\frac{1}{n+5}-\frac{1}{n+3}\right]+\sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+3}\right]\right)$$

Now $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)=\frac{5}{6}$ (A)

$\sum_{n=1}^{\infty}\left(\frac{1}{n+5}-\frac{1}{n+3}\right)=\frac{-9}{20}$ (B)

and $\frac{1}{8}\left(\frac{5}{6}-\frac{9}{20}\right)=\frac{23}{480}$.

For A

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)=\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{7}+\frac{1}{6}-\frac{1}{8}+\dots$$

$$=\tfrac{1}{2}+\tfrac{1}{3}\left(-\tfrac{1}{4}+\frac{1}{5}+\frac{1}{4}-\tfrac{1}{6}+\tfrac{1}{5}-\frac{1}{7}+\dots\right)$$

$=\frac{5}{6}$ as all other terms cancel.

For B

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+5}-\frac{1}{n+3}\right)=\frac{1}{6}-\frac{1}{4}+\frac{1}{7}-\frac{1}{5}+\frac{1}{8}-\frac{1}{6}+\frac{1}{9}-\frac{1}{7}+\frac{1}{10}-\frac{1}{8}+\dots$$

$$=-\frac{1}{4}-\frac{1}{5}+\left(\frac{1}{6}+\frac{1}{7}+\tfrac{1}{8}-\frac{1}{6}+\frac{1}{9}-\frac{1}{7}+\frac{1}{10}-\tfrac{1}{8}+\dots\right)$$

$=-\frac{9}{20}$ as all other terms cancel.

(7) $\sum_{n=1}^{\infty}\frac{3n-2}{n(n+1)(n+2)}$ Now $\frac{3n-2}{n(n+1)(n+2)}=\frac{5}{n+1}-\frac{1}{n}-\frac{4}{n+2}$

$$=\frac{1}{n+1}-\frac{1}{n}+4\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

Now $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right)=\frac{1}{2}-1+\frac{1}{3}-\frac{1}{2}+\frac{1}{4}-\frac{1}{3}+\dots$

$=-1$,

Now $\sum_{n=1}^{\infty}\frac{1}{n+1}-\frac{1}{n+2}=\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\dots$

$=\tfrac{1}{2}$

So $\sum_{n=1}^{\infty}\frac{3n-2}{n(n+1)(n+2)}=-1+4\times\frac{1}{2}=-1+2=1$.

(8) $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \longrightarrow S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Let S_{3N} be the nth partial sum of the rearranged series. Then we see

$$S_{3N} = 1 - \frac{1}{2} - \frac{1}{4} + 1/3 - \cdots + \frac{1}{2N-1} - \frac{1}{4N-2} - \frac{1}{4N}$$

$$= \left(1 + 1/3 + \frac{1}{2N-1}\right) - \left(\frac{1}{2} + 1/6 + \cdots + \frac{1}{4N-2}\right) - \left(1/4 + 1/8 + \cdots + \frac{1}{4N}\right)$$

$$= \left(1 + 1/3 + \cdots + \frac{1}{2N-1}\right) - \frac{1}{2}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2N-1}\right) - \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2N}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2N-1} - \frac{1}{2N}\right) \to \frac{1}{2} S \text{ as } N \to \infty.$$