Tutorial 5 Solutions RA

Q1 (i) $\lim\limits_{x\to 2} \dfrac{x}{x+3} = \dfrac{2}{5}$. To show this write

$$\left|\frac{x}{x+3} - \frac{2}{5}\right| = \left|\frac{5x - 2(x+3)}{5(x+3)}\right| = \left|\frac{3(x-2)}{5(x+3)}\right|$$

Now since $x \to 2$, we may assume $x>0$.

Let $\varepsilon > 0$. $\left|\dfrac{3}{5(x+3)}\right| \leq \dfrac{3}{5.3} = \dfrac{1}{5}$.

Choose $\delta > 0$ by letting $\delta = 5\varepsilon$.

Then $|x-2| < \delta \Rightarrow \left|\dfrac{x}{x+3} - \dfrac{2}{5}\right| < 5.\dfrac{1}{5}\varepsilon = \varepsilon$

So $\dfrac{x}{x+3} \to \dfrac{2}{5}$

(ii) $\lim\limits_{x\to\infty} (\sqrt{x^2+4} - x) = \lim\limits_{x\to\infty} (\sqrt{x^2+4} - x)\left(\dfrac{\sqrt{x^2+4} + x}{\sqrt{x^2+4} + x}\right)$

$$\lim_{x\to\infty} = \frac{4}{\sqrt{x^2+4} + x} = 0$$

To show this from definition we have to find $M>0$ such that $x > M \Rightarrow \left|\dfrac{4}{\sqrt{x^2+4}+x} - 0\right| < \varepsilon$.

Since $\sqrt{x^2+4} + x > 2x$,

$$\frac{4}{\sqrt{x^2+4}+x} < \frac{4}{2x} = \frac{2}{x}$$

If $\dfrac{2}{x} < \varepsilon$, $x > \dfrac{2}{\varepsilon}$. Let $M \geq \dfrac{2}{\varepsilon}$.

Then $x > M \Rightarrow \left|\sqrt{x^2+4} - x\right| < \dfrac{2}{x} < \varepsilon$.

So $\lim\limits_{x\to\infty} (\sqrt{x^2+4} - x) = 0$

(iii) $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{x} = \lim\limits_{x\to 0} \dfrac{2\sin x\cos x}{x} = \lim\limits_{x\to 0} \dfrac{2\sin x}{x} \lim\limits_{x\to 0} \cos x$

$= 2$

(iv) $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4} = \lim\limits_{x\to 2} \dfrac{\cancel{x-2}}{(\cancel{x-2})(x+2)} = \lim\limits_{x\to 2} \dfrac{1}{x+2} = \dfrac{1}{4}$

(2)(i) $\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a} cf(x) = cL$. Assume $c \neq 0$ ($c=0$ is trivial)

Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x)-L| < \frac{\varepsilon}{|c|}$.

Then $|x-a| < \delta \Rightarrow |cf(x) - cL| < |c| \frac{\varepsilon}{|c|} = \varepsilon$

So $cf(x) \to cL$

(ii) $f(x) \to L$, $g(x) \to M$. Choose $\delta_1 > 0$ such that $|x-a| < \delta_1 \Rightarrow |f(x)-L| < \varepsilon/2$

Choose δ_2 such that $|x-a| < \delta_2 \Rightarrow |g(x)-L| < \frac{\varepsilon}{2}$

Take $\delta = \min(\delta_1, \delta_2)$. Then

$$|x-a| < \delta \Rightarrow |f(x)+g(x)-L-M| \leq |f(x)-L| + |g(x)-M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

(iii) $f(x) \to L$, $g(x) \to M$ so $g(x)f(x) \to LM$. Take $L \neq 0$, $M \neq 0$

We have

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM|$$
$$= |f(x)(g(x) - M) + M(f(x) - L)|$$
$$\leq |f(x)||g(x) - M| + |M||f(x) - L|$$

Choose δ_1 such that $|x-a| < \delta_1 \Rightarrow |f(x)-L| < \frac{\varepsilon}{2|M|}$

Now x is near a, so $|f(x)|$ is near $|L|$

Pick δ_2 such that $|x-a| < \delta_2 \Rightarrow |g(x) - L| < \frac{\varepsilon}{2(1+|L|)}$

Now if $\delta_2 < \delta_1$, $|f(x) - L| < \varepsilon$

So $-\varepsilon < f(x) - L < \varepsilon$

And $|f(x)| < \varepsilon + L \leq \varepsilon + |L|$. If $\varepsilon < 1$ we have $|f(x)| < 1 + |L|$

Take $\delta = \min(\delta_1, \delta_2)$. $|x-a| < \delta$

$$\Rightarrow |f(x)g(x) - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$$
$$< \frac{\varepsilon(1+|L|)}{2(1+|L|)} + M\frac{\varepsilon}{2|M|}$$
$$= \varepsilon$$

(iv) Assume $g(x) > k > 0$ and $M \neq 0$. Then we

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| = \left|\frac{Mf(x) - Lg(x)}{g(x)\, M}\right|$$

$$\leq \frac{1}{|g(x) M|} |Mf(x) - ML + ML - Lg(x)|$$

$$\leq \frac{1}{kM} |M(f(x) - L) + L(M - g(x))|$$

$$\leq \frac{1}{|k||M|}\left(|M||f(x) - L| + |L||M - g(x)|\right)$$

Pick $\delta_1 > 0$ st $|x-a| < \delta_1 \Rightarrow |f(x) - L| < |k| \frac{\varepsilon}{2}$

" $\delta_2 > 0$ " " $< \delta_2 \Rightarrow |M - g(x)| < \frac{|k| M}{2|L|} \varepsilon$

If $\delta = \min(\delta_1, \delta_2)$

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| \leq \frac{1}{|k| M}\left(|M||f(x) - L| + |L||M - g(x)|\right)$$

$$< \frac{1}{|k|} |k| \frac{\varepsilon}{2} + \frac{|k||M||L|}{2|k||M|} \varepsilon = \varepsilon.$$

We here assumed $L \neq 0$. If $L = 0$

The term $|L||M - g(x)|$ disappears.

(3). We did $f(x) = x^2$ in class(*). So let us do $f(x) = x^3$ instead. We let $x_n \to x$. So we can find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |x_n - x| < \frac{\varepsilon}{K}$ for some K.

Now $|x_n^3 - x^3| = |x_n - x||x_n^2 + x_n x + x^2|$

$$\leq |x_n - x|(|x_n^2| + |x_n x| + |x|^2)$$

Assume $x = 0$ otherwise it is trivial $(0 = 0)$

$\{x_n\}$ is convergent, so it is bounded. Let $|x_n| \leq M$.

Then take $K = M^2 + M|x| + |x|^2$.

Thus $n \geq N \Rightarrow |x_n^3 - x^3| < |x_n - x| K < K \frac{\varepsilon}{K} = \varepsilon$

So $x_n^3 \to x^3$ and x^3 is therefore continuous

(*) ~~I forgot it was in the title~~

For $\sin x$ we have $|\sin x| \le |x|$
and we use the trig identity

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

So
$$|\sin x - \sin y| = 2\left|\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right|$$
$$\le 2\left|\sin\left(\frac{x-y}{2}\right)\right|$$
$$\le 2\left|\frac{x-y}{2}\right| = |x-y|$$

So if $|x-y| < \varepsilon$ Then $|\sin x - \sin y| < \varepsilon$ and thus $\sin x$ is uniformly continuous on $\mathbb{R}$
Note $\cos x$ is also uniformly continuous on $\mathbb{R}$.

(4) It is enough to show that x^n is continuous for any positive integer n.
We use
$$x^n - y^n = (x-y)(x^{n-1} + x^{n-1}y + x^{n-2}y^2 + \cdots + y^n)$$
If $x_k \to x$
$$|x_k^n - x^n| \le K|x-y|$$
for $K = |x^n| + \cdots + M^{n-1}$
where $|x_n| \le M$ all n. The proof then is the same as for Q 3.

(4) Linear Combinations of continuous functions are continuous. So it suffices to show that x^n is continuous on $\mathbb{R}$ for $n = 0, 1, 2, 3, \ldots$

Now
$$|x^n - y^n| = |x-y|\,|x^{n-1} + x^{n-2}y + \ldots + y^{n-1}|$$
Now for any finite x, y, $\exists\, M > 0$, $M < \infty$ such that
$$|x^{n-1} + \cdots + y^{n-1}| \le M.$$
Let $\varepsilon > 0$. Choose x, y such that $|x-y| < \frac{\varepsilon}{M}$.

then $|x-y| < \delta$, $\delta = \varepsilon/M$
$$\Rightarrow |x^n - y^n| < \frac{\varepsilon}{M} M = \varepsilon$$

So x^n is continuous.

(5) $f(r) = r^2$, $r \in I \cap \mathbb{Q}$. f is continuous
Let $x \in I$. If $x \in \mathbb{Q}$, $f(x) = x^2$.
Suppose that x is irrational. Then we can find a sequence of rationals $r_n \in I$ such that $r_n \to x$.
By continuity $f(r_n) \to f(x)$
So $r_n^2 \to f(x)$ or $f(x) = \lim_{n\to\infty} r_n^2$
$$= x^2$$
by elementary properties of limits.

(6) Continuous functions have the intermediate value property. So if $f(a) < 0$, $f(b) > 0$ then there exists $\xi \in (a, b)$ such that $f(\xi) = 0$.
Let $P_{2n+1}(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \ldots + a_0$
$$= a_{2n+1}x^{2n+1} + q_{2n}(x)$$
$a_{2n+1} \ne 0$

Now $\lim_{|x| \to \infty} \frac{q_{2n}(x)}{a_{2n+1} x^{2n+1}} = 0$.

Without loss of generality, assume $a_{2n+1} > 0$
So we can find x large enough that

$$x^{2n+1} > \left| \frac{q_{2n}(x)}{a_{2n+1}} \right|$$

For such $\bar{x}$, $p_{2n+1}(x) > 0$.
Conversely we can find $\underline{x} < 0$
such that $|-\underline{x}|^{2n+1} > \left| \frac{q_{2n}(\underline{x})}{a_{2n+1}} \right|$

which means $p_{2n+1}(x) < 0$. So by the intermediate property, there is an $\xi \in \mathbb{R}$ such that $p_{2n+1}(\xi) = 0$.

Note that if P_{2n} is even, this is not necessarily true

7) For every $x \in I$ there is a $y \in I$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$.
Since $y \in I$ $\exists$ $y_1 \in I$ with

$$|f(y_1)| \leq \frac{1}{2}|f(y)| \leq \frac{1}{2^2}|f(x)|$$

we can also find $y_2 \in I$ with $|f(y_2)| \leq \frac{1}{2}|f(y_1)| \leq \frac{1}{2^3}|f(x)|$

Thus there is a sequence $\{y_n\}$ such that $|f(y_n)| \leq \frac{1}{2^{n+1}}|f(x)|$.

Now $\{y_n\} \in [a,b)$, so it is bounded.
Clearly $\lim_{n \to \infty} |f(y_n)| = 0$.

Since $\{y_n\}$ is bounded, it has a convergent

subsequence $\{y_{n_k}\}$. Let $y_{n_k} \to \xi \in I$.
Then

$$\lim_{k\to\infty} |f(y_{n_k})| = |f(\lim_{k\to\infty} y_{n_k})|$$
$$= |f(\xi)| = 0$$

since f is continuous. Thus
$$f(\xi) = 0$$

(8) This is a simple version of Brouwer's fixed point theorem.
Now $f: [a,b] \to [a,b]$. So the image of $[a,b]$ under f is a subset of $[a,b]$.
So $f(a) \geq a$, $f(b) \leq b$
Let $g(x) = f(x) - x$, $x \in [a,b]$. This is continuous. But $g(a) \geq 0$ and $g(b) \leq 0$.
So $\exists\ \xi \in [a,b]$ with $g(\xi) = \xi$
or $f(\xi) = \xi$

(9) Let $J = f(I) = \{f(x) : x \in I\}$. We show that J is also an interval. We have to prove that if $y_1, y_2 \in J$ and $y_1 \leq \lambda \leq y_2$, then $\lambda \in J$.
Now
$$S = \{x : f(x) \leq \lambda\},\ T = \{x : f(x) \geq \lambda\}$$
are non empty. Every point in I is in either S or T. (There are no other options).
Consider T. We can find $s \in S$, which is at zero distance from T. (See Canvas).
We can find a sequence $\{t_n\} \subset T$, with $t_n \to s$ as $n \to \infty$. By continuity
$f(t_n) \to f(s)$. $f(t_n) \geq \lambda$ all n, so $f(s) \geq \lambda$. But $s \in S$, so $f(s) \leq \lambda$.
$$\therefore f(s) = \lambda.$$
The other way round is similar

(10) Suppose ξ is such that $f(\xi) > 0$.
Since $f(x) \to 0$ as $x \to \infty$, we can find $b > \xi$, such that for $x > b$ $|f(x)| < |f(\xi)|$.
Since $f(x) \to 0$ as $x \to -\infty$, we can find $a < \xi$, such that for any $x < a$, $|f(x)| < f(\xi)$. So f attains a maximum on $[a,b]$. Suppose that $f(\eta) \geq f(x)$, $x \in [a,b]$. Then $f(\eta) \geq f(\xi) > f(x)$, $x \notin [a,b]$.
Hence f attains a maximum on $\mathbb{R}$.
If we can find ξ such that $f(\xi) < 0$, the existence of a minimum is proved a similar way.
The case $f(x) = 0$ all x is clearly trivial.