Tutorial 7 Solutions. RA

(1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad a_n = \frac{x^n}{n!}$

(i) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!}\right| = \frac{|x|}{n+1}$

$\lim_{n\to\infty} \frac{|x|}{n+1} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 \quad \text{all } x \in \mathbb{R}$

(ii) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(ii) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+3}}{(2n+3)!} \Big/ \left|\frac{x^{2n+1}}{(2n+1)!}\right| = \frac{|x|^2}{(2n+3)(2n+2)}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|^2 \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0, \quad \text{all } x \in \mathbb{R}$

(iii) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{(2n+2)!} \Big/ \frac{x^{2n}}{(2n)!}\right| = \frac{|x|^2}{(2n+2)(2n+1)}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 \quad \text{all } x \in \mathbb{R}$

(2) $f(x) = \sin x \qquad a = \pi/2$

$f(a) = \sin\left(\frac{\pi}{2}\right) = 1$

$f'(a) = \cos\left(\frac{\pi}{2}\right) = 0$

$f''(a) = -\sin\left(\frac{\pi}{2}\right) = -1$

$f'''(a) = -\cos\left(\frac{\pi}{2}\right) = 0$

$f^{(iv)}(a) = \sin\left(\frac{\pi}{2}\right) = 1 \quad \text{etc}$

$\therefore \sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} + -$

By differentiation

$\cos x = -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} - -$

So the series for $\sin x$ is in even power of $(x-\frac{\pi}{2})$ and the series for $\cos x$ is in odd powers of $(x-\pi/2)$ — the opposite of what happens if $a=0$.

$f(x) = (1+x)^{\alpha}$ Take $a=0$

$f(0) = 1$, $f'(x) = \alpha(1+x)^{\alpha-1}$

$f'(0) = \alpha$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$, $f''(0) = \alpha(\alpha-1)$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$

$f'''(0) = \alpha(\alpha-1)(\alpha-2)$

In general

$f^{(n)}(0) = \alpha(\alpha-1)\dots(\alpha-n+1)$

So the Taylor series is

$$1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2} + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

This is the Binomial series

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha(\alpha-1)\dots(\alpha-n)x^{n+1}}{(n+1)!} \Big/ \frac{\alpha(\alpha-1)\dots(\alpha-n+1)x^n}{n!}\right|$$

$$= |x|\left|\frac{(\alpha-n)}{n+1}\right| \to |x| \text{ as } n \to \infty$$

So we have convergence if $|x|<1$.

(4) Take $\alpha = 1/2$, $x = 1/2$

$$(3/2)^{1/2} = \sqrt{3/2} \simeq 1 + \frac{1}{4} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}\left(\frac{1}{2}\right)^2 = \frac{39}{32} = 1.21875$$

By calculator $\sqrt{\frac{3}{2}} = 1.22474\dots$

So 3 terms gives quite good accuracy

Four terms gives

$$\sqrt{3/2} \simeq 1 + \frac{1}{4} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{1}{2}\right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(1/2-2)}{3!}\left(\frac{1}{2}\right)^3 = \frac{157}{128} = 1.21875.$$

(5) $f(x) = \dfrac{x}{(1+x^2)^2}$.

The geometric series is

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \dots \qquad |r|<1.$$

take $r = x^2$. Then

$$f(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - .$$

Differentiate both sides

$$f'(x) = \frac{-2x}{(1+x^2)^2} = -2x + 4x^3 - 6x^5 + 8x^7 - -$$

Or $$\frac{x}{(1+x^2)^2} = x - 2x^3 + 3x^5 - 4x^7 \ldots \ldots$$

The series has the same radius of convergence as that of $f(x)$ i.e. 1

(6) $f(x) = \ln(1+x)$.

$$f'(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

So $$f(x) = \int_0^x \frac{1}{1+t}\,dt \qquad (*)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots$$

It has the same radius of convergence as $\frac{1}{1+x}$ i.e. 1

(*) We can also use $f''(x) = \frac{-1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$ etc

(7) $$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}. \qquad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{\sqrt{n+1}} \Big/ \frac{x^n}{\sqrt{n}}\right|$$

$$= |x| \sqrt{\frac{n}{n+1}} = |x| \sqrt{1 - \frac{1}{n+1}}$$

$\rightarrow |x|$ as $n \to \infty$. So radius of convergence is 1.

$$\sum_{n=1}^{\infty} \frac{x^2}{n^2+1}.$$ Converges for all x,

This is obvious

(8) $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(0) = \lim_{x \to 0} \frac{e^{-1/x^2} - 0}{x - 0} = 0.$

So $f'(0) = 0$

$f'(x) = \begin{cases} \frac{2}{x^3} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f''(0) = \lim_{x \to 0} \frac{\frac{2}{x^3} e^{-1/x^2} - 0}{x - 0} = 0$

$\therefore f''(x) = \begin{cases} \left(\frac{4}{x^6} - \frac{6}{x^4}\right) e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'''(0) = 0$ by a similar calculation

Infact $f^{(n)}(0) = 0$ all n.

So Taylor's formula with $a = 0$ gives the Taylor expansion Tf as

$Tf(x) = 0 + 0 + 0 + \cdots$
$= 0$

This equals f only at $x = 0$.

9) $\sum_{n=0}^{\infty} n^2 x^n \qquad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 x^{n+1}}{n^2 x^n}\right|$

$= |x| \left(1 + \frac{1}{n}\right)^2 \to |x|$ as $n \to \infty$

So series converges for $|x| < 1$.

Now
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Then $\frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

So $\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

Differentiating again we have

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{x+1}{(1-x)^3}$$

Thus $\displaystyle\sum_{n=1}^{\infty} n^2 x^{n-1} = \sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$.

(10) $\displaystyle y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

So $a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$\therefore$ $a_1 = a_0$ $\quad a_1 = 2a_2$

or $\quad a_2 = \frac{1}{2} a_1 = \frac{1}{2} a_0$

$3a_3 = a_2 \quad \therefore a_3 = \frac{1}{3} a_2 = \frac{1}{1\times2\times3} a_0$

$4a_4 = a_3 \Rightarrow a_4 = \frac{1}{4} a_3 = \frac{1}{1\times2\times3\times4} a_0$

In general $a_n = \frac{1}{n!} a_0$

$\therefore$ $\displaystyle y = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ or $y = a_0 e^x$.