Tutorial 8 Solutions

1) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$ by L'Hôpital

(ii) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$

iii) $\lim_{z\to -1} \frac{z+1}{z^5+1} = \lim_{z\to -1} \frac{1}{5z^4} = \frac{1}{5}$.

(iv) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = \lim_{x\to 0} \frac{1}{\cos^2 x} = 1$

v) $\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{4x^2-\pi^2} = \lim_{x\to \frac{\pi}{2}} \frac{\sin x}{8x} = \frac{\sin\frac{\pi}{2}}{8\frac{\pi}{2}} = \frac{1}{4\pi}$.

(2) $f(z) = \ln z$

$\frac{\ln z - \ln w}{z-w} = f'(c)$ for some c

$f'(c) = \frac{1}{c} \leq 3$ for $c \in [\frac{1}{3}, 3]$.

Thus $|\ln z - \ln w| \leq 3|z-w|$, $z, w \in [\frac{1}{3}, 3]$

3) Same as previous question.

$f(x) = \sin x$. $|f'(x)| = |\cos x| \leq 1$.

Since $\frac{\sin x - \sin y}{x-y} = f'(c)$ some c, $|\sin x - \sin y| \leq |x-y|$

4) f is continuously differentiable on $[-7,0]$ $f(-7)=3$, $f'(x)\leq 2$

$\frac{f(0)-f(-7)}{0-(-7)} = f'(c)$ some c.
≤ 2

$\therefore f(0) \leq 2.7 + f(-7) = 3$
$= 17$.

5) f is differentiable on $(1,3)$, continuous on $[1,4]$

$$\frac{f(3)-f(1)}{3-1} = f'(c)$$

So $f(3)-f(1) = 2f'(c)$, but $f'(c) \leq 2$,

$\therefore f(3)-f(1) \leq 4$

$f'(c) \geq 1$, $\therefore f(3)-f(1) \geq 2$

6) The velocity of the train is between 40 and 50 Kmh (This is a very slow train). $v = \frac{ds}{dt}$. Duration of trip is T hours. $\frac{s(T)-s(0)}{T-0} = v(c)$ $c \in [0,T]$

$s(T) = 200$

$$\frac{200}{T} = v(c)$$

$$T = \frac{200}{v(c)}$$

$$\therefore 4 \leq T \leq 5$$

You obviously do not need the MVT for this

7) $f(x) = a_n x^n + \dots + a_0$

$$f(x) = (x-a)^2(x-b)^2 p(x)$$

$$f'(x) = (x-a)^2(x-b)^2 p'(x) + 2(x-a)(x-b)^2 p'(x) + 2(x-a)^2(x-b)p(x)$$

a and b are roots. Now there exists $c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$= 0$$

So there is a third root in (a,b).

(8) Let $x > y$. Then $\exists\ c \in [x, y]$ with

$$\frac{f(x) - f(y)}{x - y} = f'(c) < 0$$

So $f(x) < f(y)$ and f is strictly decreasing.

9) Let $f(\alpha) = f(\beta) = f(\gamma) = 0$, $\alpha, \beta, \gamma \in (a, b)$ $\alpha < \beta < \gamma$

There exists $c_1 \in (\alpha, \beta)$ with

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(c_1).$$ But $f(\alpha) = f(\beta) = 0$

So $f'(c_1) = 0$. Similarly there is a $c_2 \in (\beta, \gamma)$ with

$$\frac{f(\gamma) - f(\beta)}{\gamma - \beta} = f'(c_2) = 0$$

Now apply the MVT to f' on (c_1, c_2). There exists $x_0 \in (c_1, c_2) \subset (a, b)$ such that

$$\frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = f''(x_0)$$

But $f'(c_1) = f'(c_2) = 0$. So $f''(x_0) = 0$.

10) This is a powerful method of solving differential equations.

$$y = \sum_{n=0}^{\infty} a_n x^n. \quad \text{So} \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Hence $$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = x \sum_{n=0}^{\infty} a_n x^n$$

or $$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1}.$$

Now $\sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1}$

But $\sum_{n=2}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1}$.

We lowered the starting value from $n=2$ to $n=0$ so we raised the n's inside the sum by 2. Check that both sides are the same.

Hence $y' = xy$ is the same as

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} = \sum_{n=0}^{\infty} a_n x^{n+1}.$$

So we must have $a_1 = 0$ as there is no constant term on the right side. (all terms are multiplied by x).

Thus $\sum_{n=0}^{\infty} \left[(n+2) a_{n+2} - a_n\right] x^{n+1} = 0$.

This can only be true if

$$(n+2) a_{n+2} = a_n.$$

or $a_{n+2} = \dfrac{a_n}{n+2}$.

If $n=1$, $a_3 = \dfrac{a_1}{3} = 0$

$n=3$ gives $a_5 = \dfrac{a_3}{5} = 0$. The pattern is

that $a_1 = a_3 = a_5 = a_7 = \ldots = 0$.

So there are no <u>odd</u> powers.

If $n=0$, $a_2 = \frac{a_0}{2}$.

$n=2$ gives $a_4 = \frac{a_2}{2\times4} = \frac{a_0}{2\times2\times2}$.

$n=4$ gives $a_6 = \frac{a_4}{6} = \frac{a_0}{2^3(1\times2\times3)}$

$n=6$ gives $a_8 = \frac{a_6}{8} = \frac{a_0}{2^4(1\times2\times3\times4)}$

In general

$$a_{2k} = \frac{a_0}{2^k k!}, \quad k=0,1,2,\ldots$$

Thus $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$

(Split into odd and even parts)

$$\therefore y = \sum_{k=0}^{\infty} \frac{a_0}{k!\,2^k} x^{2k} = \sum_{k=0}^{\infty} \frac{a_0}{k!}\left(\frac{x^2}{2}\right)^k$$

$$= a_0 e^{\frac{x^2}{2}} \quad \left(e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},\ \text{put } z = \frac{x^2}{2}\right)$$

Since $y(0) = 1 = a_0$, solution is

$$y = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^2}{2}\right)^k = e^{x^2/2}.$$