Real Analysis 35007

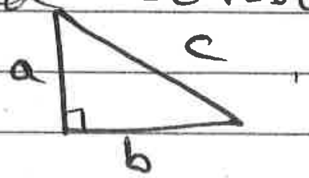
Workshop I notes Aug 8 ①

Introduction. Theorems and proofs. The structure of Analysis.

Let us start with the following.

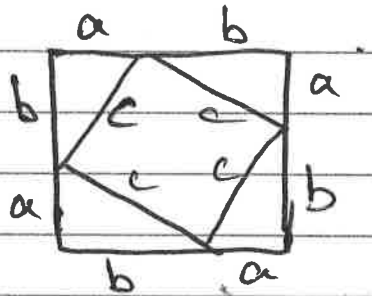
Theorem (Pythagoras?)

Let $a, b, c > 0$ and consider the right triangle



Then $a^2 + b^2 = c^2$

Proof Consider the square

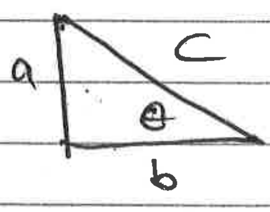


Area of big square is

$(a+b)(a+b)$
$= a^2 + 2ab + b^2$
$= c^2 + 4 \frac{1}{2} ab$
$= c^2 + 2ab$

$\therefore$ $\boxed{a^2 + b^2 = c^2}$

Corollary Let θ be an angle. Then $\cos^2\theta + \sin^2\theta = 1$



$\cos\theta = \frac{b}{c}$, $\sin\theta = \frac{a}{c}$

So $\left(\frac{a}{b}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$.

since $a^2 + b^2 = c^2$.

This is the standard pattern for mathematical writing. We have a Theorem followed by a proof. Sometimes there is a corollary.

Our most important tool is the <u>triangle inequality</u>.

<u>Theorem</u> Let $a, b \in \mathbb{R}$. Let $|\cdot|$ denote the <u>absolute value</u>

$$|x| = +\sqrt{x^2}.$$

Then $|a+b| \le |a| + |b|$.

<u>Proof</u> We have

$$|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2$$
$$\le |a|^2 + 2|ab| + |b|^2$$
$$= (|a|+|b|)^2.$$

So $|a+b| \le |a| + |b|$,

we used $ab \le |ab|$

<u>Archimedian Property</u> There is no largest integer.

<u>Upper and lower bounds</u> If A is a set of real numbers and u is a real number such that

$$u \ge x \text{ for all } x \in A$$

Then <u>u is</u> an upper bound for A.

If l is a real number and $l \le x$ for all $x \in A$, then l is a <u>lower bound</u>

Let us consider an example

<u>Example</u> $A = [1, 5]$

Then 7 is an upper bound and 0 is a lower bound

If u is an upper bound for A and for any other upper bound U $u \le U$, then u is the <u>least upper bound</u>

Similarly we have the greatest lower bound. If l is a lower bound and for any other lower bound L, $l \geq L$, l is the greatest lower bound

Definition If a set $A \subseteq \mathbb{R}$ has a least upper bound u, we call u the supremum of A. Usually we write $\sup A = u$. If l is the greatest lower bound we call it the infimum of A. We write $\inf A = l$

Example $A = (0,1)$
$\sup A = 1$. It is the smallest number bigger than everything in A.
Similarly $\inf A = 0$

Axiom 1 Every non empty set of real numbers that is bounded above has a least upper bound.
Theorem Every non empty set of real numbers bounded below has a greatest lower bound
Proof (Notes)
Idea. Consider a set A define $-A = \{-x, x \in A\}$. If A has a lower bound $-A$ has an upper bound
Let $A = [0,2]$, $-A = [-2,0]$
$\sup(-A) = 0$. Then $-\sup(-A) = \inf A$
$= 0$

<u>Theorem 1.14</u> Assume that $\sup A$ exists where $A \subset \mathbb{R}$. Then for $\varepsilon > 0$ we can find an x such that

$$\sup A - \varepsilon < x \leq \sup A$$

<u>Proof (Notes)</u>

<u>Sequences and Limits</u> We are all familiar with the idea of a limit. In calculus the derivative is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

But what does this actually mean?

<u>Definition(*)</u> Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $a \in \mathbb{R}$. We say that a_n converges to a if given any real number $\varepsilon > 0$, we can find a natural number $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|a_n - a| < \varepsilon.$$

We write $a_n \to a$ or $\lim_{n \to \infty} a_n = a$.

This means that we can make the difference between the terms of the sequence as small as we desire by going out far enough. Let $a_n \to a$

$$a_1, a_2, \ldots, \underbrace{a_N, a_{N+1}, a_{N+2}, \ldots}$$

$|a_N - a| < \varepsilon$, $|a_{N+1} - a| < \varepsilon$

$|a_{N+2} - a| < \varepsilon$ etc

(*) This is the most important part of the lecture

Example 1 $a_n = \frac{1}{n}$. Let $\varepsilon = \frac{1}{10}$

Let $a = 0$

Then $|a_n - 0| = |\frac{1}{n}| = \frac{1}{n}$

If $|a_n - 0| = \frac{1}{n} < \frac{1}{10}$

If $N = 11$, $n \geq 11 \Rightarrow |a_n - 0| < \frac{1}{10}$

Now let ε be arbitrary

We want $\left|\frac{1}{n}\right| < \varepsilon$

If $n \geq N \in \mathbb{N}$, and $N > \frac{1}{\varepsilon}$

Then $\frac{1}{n} < \frac{1}{N} < \varepsilon$.

To prove that $a_n \to a$ we solve an inequality. This is the point, You need to learn this.

Example 2 $a_n = \frac{n^2 - 4}{n^2 + 4}$, $n = 1, 2, 3$

Let $a = 1$

$$\left|\frac{n^2-4}{n^2+4} - 1\right| = \left|\frac{n^2 - 4 - (n^2+4)}{n^2+4}\right|$$

$$= \left|\frac{-8}{n^2+4}\right| = \frac{8}{n^2+4}.$$

When is this $< \varepsilon$?

$$\frac{8}{n^2+4} < \varepsilon \Rightarrow n^2 + 4 > \frac{8}{\varepsilon}$$

$$n^2 > \frac{8}{\varepsilon} - 4$$

Take $N \in \mathbb{N}$, st. $N > \sqrt{\frac{8}{\varepsilon} - 4}$,

This is messy, but correct. A more elegant approach is to observe that $n^2 + 4 > n$ for all n,

So $\frac{1}{n^2+4} < \frac{1}{n}$, or $\frac{8}{n^2+4} < \frac{8}{n}$

So if $\frac{8}{n} < \varepsilon$, then $\frac{8}{n^2+4} < \varepsilon$. Take $N > \frac{8}{\varepsilon}$,

and for $n \geq N$ $|a_n - 1| < \varepsilon$.

Theorem The limit of a convergent sequence is unique.

Proof Let $\{x_n\}$ be a convergent sequence. Suppose that $x_n \to x$ and $x_n \to y$.
Let $\varepsilon > 0$.

Then
$$|x-y| = |x - x_n + x_n - y|$$
$$\leq |x - x_n| + |x_n - y| \quad \text{(triangle inequality)}$$
$$= |x_n - x| + |x_n - y|$$

Choose N_1 such that $n \geq N_1$
$$\Rightarrow |x_n - x| < \frac{\varepsilon}{2}$$
Choose N_2 such that $n \geq N_2$
$$\Rightarrow |x_n - y| < \frac{\varepsilon}{2}.$$

Then let $N = \max\{N_1, N_2\}$

Hence $|x-y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

But $\varepsilon > 0$ is arbitrary and $|x-y| \geq 0$. So $|x-y| < \varepsilon$ for all $\varepsilon > 0$
$$\Rightarrow |x-y| = 0. \text{ Or } x = y.$$

Theorem Every convergent sequence is bounded.

Proof Let $\{x_n\}$ be convergent, lim. x

Choose N s.t. $n \geq N \Rightarrow |x_n - x| < 1$

Let $M = \max\{|x_1|, \ldots, |x_{N-1}|, 1+|x|\} < \infty$.

If $1 \leq n \leq N-1$ clearly $|x_n| \leq M$

If $n \geq N$, $|x_n| = |x_n - x + x| \leq |x_n - x| + |x|$
$$< 1 + |x| \leq M$$

So $|x_n|$ is bounded by M.