Tutorial One Solutions

Inf and Sup

(1) Let $B = \sup S$. Then for all $x \in S$
$$x \le B.$$
Let $T = \{\xi x ; x \in S\}$. Since $\xi > 0$
$$\xi x \le \xi B, \text{ for any } x \in S.$$
Hence T is bounded above by ξB.
So T has a least upper bound, Call this C.
We show $C = \xi B$. Clearly $C \le \xi B$.
Now reverse roles of S and T. C is the smallest number such that for any $y \in T$, $y \le C$. Since $\xi > 0$
$$\frac{1}{\xi} y \le \frac{1}{\xi} C$$
for any $y \in T$. But $S = \{\frac{y}{\xi} : y \in T\}$. Hence $\frac{1}{\xi} C$ is an upper bound for S. But B is the smallest upper bound for S. Thus
$$\frac{1}{\xi} C \ge B \quad . \quad \text{or } C \ge \xi B.$$

So $\quad C \ge \xi B, \quad C \le \xi B.$

$\therefore \quad C = \xi B$ and Thus
$$\sup_{x \in S}(\xi x) = \xi \sup_{x \in S} x.$$

(2) S is bounded above. Let $\sup = B$.
So for all $x \in S$, $x \le B$. But $S_0 \subseteq S$, so for $y \in S_0$
$$y \le \sup S_0 \le B \le \sup S$$

(3) Let $B = \sup S$, $T = \{x + \xi : x \in S\}$. Since $x \le B$ for all $x \in S$, $\xi + x \le \xi + B$ for $x \in S$.
Hence $\xi + B$ is an upper bound for T. Let C be the smallest upper bound for T, then

$C \le \xi + B$. Now $y \le C$ for any $y \in T$,
So that
$$y - \xi \le C - \xi \quad \text{for any } y \in T.$$
Since $S = \{y - \xi : y \in T\}$ we see that $C - \xi$ is an upper bound for S.
$$\therefore \; B \le C - \xi.$$
$$\therefore \; B + \xi \le C.$$
$$\therefore \; B + \xi \ge C \quad \text{and} \quad B + \xi \le C.$$
$$\text{So } C = B + \xi.$$

(4).(i) Let $D = \{|\xi - x| : x \in S\}$. This has zero as a lower bound, (It cannot have any negative elements). If $\xi \in S$, then the minimum is 0.

ii) Let $\xi = \sup S$. Then $|\xi - x| = \xi - x$ for each $x \in S$. We show that no $h > 0$ is a lower bound for $D = \{\xi - x : x \in S\}$. Suppose not. then we can find an $h > 0$ such that
$\xi - x \ge h$ for all $x \in S$. But then $x \le \xi - h$ for all $x \in S$, hence $\xi - h$ is an upper bound for S, smaller than the least upper bound. A contradiction. For the second case ($\xi = \inf S$), the proof is similar, with $d(-\xi, T)$, $T = \{-x : x \in S\}$

ii) Since I is an interval, $\xi \notin I \Rightarrow \xi$ is an upper or lower bound for I. Suppose it is an upper bound. Let B be the least upper bound of I. Then $B \in I$ because I is closed.
Given any $x \in I$ $\;|\xi - x| = \xi - x = \xi - B + B - x$
$$= \xi - B + |B - x|$$
Hence $\inf\limits_{x \in B} |\xi - x| = \xi - B + \inf\limits_{x \in B} |B - x|$ (Why?)

and therefore $d(\xi, S) = \xi - \beta + d(\beta, S)$. But $\xi - \beta \geq 0$; $d(\beta, S) \geq 0$ and $d(\xi, S) = 0$. It follows that $\xi = \beta$ and hence $\xi \in \bar{I}$.

If ξ is a lower bound, the proof is similar.

Limits

(5) $\lim_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2}$. Let $\varepsilon > 0$ then

$$\left|\frac{n}{2n+4} - \frac{1}{2}\right| = \left|\frac{2n-(2n+4)}{2(2n+4)}\right| = \frac{2}{|2n+4|}$$

So $\left|\frac{n}{2n+4} - \frac{1}{2}\right| < \varepsilon \Rightarrow \frac{1}{|2n+4|} < \frac{\varepsilon}{2}$

Since $n > 0$ $\quad 2n+4 > \frac{2}{\varepsilon}$

or $\quad n > \frac{1}{2}\left(\frac{2}{\varepsilon} - 4\right) = \frac{1}{\varepsilon} - 2.$

Choose the smallest integer N such that $N > 0$ and $N \geq \frac{1}{\varepsilon} - 2$.

Then for all $n \geq N$, $\left|\frac{n}{2n+4} - \frac{1}{2}\right| < \varepsilon \Rightarrow$
$\lim_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2}$.

(6) $\lim_{n\to\infty} \left(\frac{2n+1}{3n+2}\right) = \frac{2}{3}$ Let $\varepsilon > 0$. Then

$$\left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| = \left|\frac{3(2n+1) - 2(3n+2)}{3(3n+2)}\right|$$

$$= \left|\frac{3-4}{3(3n+2)}\right| = \frac{1}{3(3n+2)} < \varepsilon$$

$\Rightarrow 3(3n+2) > \frac{1}{\varepsilon}$. So $3n+2 > \frac{1}{3\varepsilon}$

or $3n > \frac{1}{3\varepsilon} - 2$

Hence $n > \frac{1}{9\varepsilon} - \frac{2}{3}$.

Choose $N > 0$, such that N is an integer and $N \geq \frac{1}{9\varepsilon} - 2/3$.

Thus $n \geq N \Rightarrow \left|\frac{2n+1}{3n+2} - \frac{2}{3}\right| < \varepsilon$.

(7) $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n + \frac{1}{n}}$. Now let $\varepsilon > 0$

Then $\left|\frac{1}{n+1/n}\right| < \varepsilon$ implies $n+\frac{1}{n} > \frac{1}{\varepsilon}$.

Clearly $n > 1/\varepsilon \Rightarrow n + 1/n > 1/\varepsilon$.

Choose $N \geq 1/\varepsilon$ then $n \geq N \Rightarrow \left|\frac{n}{n^2+1}\right| < \varepsilon$

So $\frac{n}{n^2+1} \longrightarrow 0.$

(8) $\lim\limits_{n\to\infty} \frac{2n^3-3n}{5n^3+4n^2-2} = \frac{2}{5}\lim\limits_{n\to\infty} \frac{n^3}{n^3+\frac{4}{5}n^2-\frac{2}{5}}$

$-3\lim\limits_{n\to\infty} \frac{n}{5n^3+4n^2-2}$

$= \frac{2}{5}$. (Proof of the individual limits is an exercise)

(9) $\lim\limits_{n\to\infty}\left(\sqrt{n^2+4}-n\right) = \lim\limits_{n\to\infty}\left(\sqrt{n^2+4}-n\right)\left(\frac{\sqrt{n^2+4}+n}{\sqrt{n^2+4}+n}\right)$

$= \lim\limits_{n\to\infty} \frac{4}{\sqrt{n^2+4}+n} = 0$

since $\frac{1}{\sqrt{n^2+4}+n} \to 0$, (Exercise)