Tutorial Two Solutions

(1) If $x=1$, $\lim_{n\to\infty} \frac{x+x^n}{1+x^n} = \frac{1+1^n}{1+1^n}$ each n

$\therefore$ limit $= 1$. If $x=-1$, sequence diverges

If $|x|>1$,

If $|x|<1$, then for some $h>0$

$$|x|^n = \frac{1}{(1+h)^n} = \frac{1}{1+nh+\frac{n(n-1)}{2}h^2+\dots+h^n} < \frac{1}{nh}$$

Now $\frac{1}{n} \to 0$ as $n\to\infty$. So $|x|^n \to 0$

and hence $x^n \to 0$ $\therefore$ For $|x|<1$

$$\lim_{n\to\infty} \frac{x+x^n}{1+x^n}$$

(2) $y_n \to 0$. So given $\varepsilon>0$ we can find $N \in \mathbb{N}$ such that $n \geq N \Rightarrow y_n < \varepsilon$.

(Clearly question implies $y_n \geq 0$)

Let $n \geq N$, then $|x_n - l| \leq y_n < \varepsilon$

So $x_n \to l$.

(3) $x_n = \left(1+\frac{1}{n}\right)^n$. The limit is actually e.

First we show sequence is bounded above

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 + \dots + \left(\frac{1}{n}\right)^n$$

$$= 1+1+\left(1-\frac{1}{n}\right)\frac{1}{2!} + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!} + \dots + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)\frac{1}{n!}$$

$$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!}$$

$$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}} \qquad (2^{n-1} \leq n!)$$

$$= 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 1+2\left(1-\left(\frac{1}{2}\right)^n\right) < 3$$

Let $a_n = 1$, $a_1 = a_2 = \ldots = a_{n-1} = 1 + \frac{1}{n-1}$. By the inequality

Then $\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{n}} \leq \frac{(n-1)\left(1+\frac{1}{n-1}\right)+1}{n} = \left(1+\frac{1}{n}\right)$

Hence
$$\left(1+\frac{1}{n-1}\right)^{n-1} \leq \left(1+\frac{1}{n}\right)^{n}, \quad n = 2, 3, \ldots$$

So the sequence is increasing and bounded above.

Thus it converges.

If we take $n = 10$, $\left(1+\frac{1}{10}\right)^{10} = 2.5937$

$n = 100$, $\left(1+\frac{1}{100}\right)^{100} = 2.7048$

$n = 1000$, $\left(1+\frac{1}{1000}\right)^{1000} = 2.7169$ etc.

(4) Let N be the smallest natural number such that $N > x$. Then for $n \geq N$
$$\frac{x^n}{n!} \leq x \cdot \frac{x}{2} \, \frac{x}{N-1} \cdot \frac{x}{N} \cdots \frac{x}{N}$$
$$= \frac{x^{N-1}}{(N-1)!}\left(\frac{x}{N}\right)^{n-N+1}.$$

Since $\frac{x}{N} < 1$, $\left(\frac{x}{N}\right)^n \to 0$ as $n \to \infty$. Now we

Question 2.

(5) Suppose that $\left(1+\frac{1}{n}\right)^{\alpha+1} |x| > 1$ for all $n \in \mathbb{N}$.

Then $\frac{1}{n} > \left(\frac{1}{|x|}\right)^{1/(\alpha+1)} - 1 > 0$

for all $n \in \mathbb{N}$ and this is a contradiction.

Hence, for some $N \in \mathbb{N}$, $\left(1+\frac{1}{N}\right)^{\alpha+1} |x| \leq 1$.

If $x \neq 0$, consider the expression

$$\left|\frac{(n+1)^{\alpha+1}x^{n+1}}{n^{\alpha+1}x^n}\right| = \left(1+\frac{1}{n}\right)^{\alpha+1}|x| \text{. If } n \geq N$$

$$|(n+1)^{\alpha+1}x^{n+1}| \leq |n^{\alpha+1}x^n|$$

It follows that for $n \geq N$ $|n^{\alpha+1}x^n| \leq |N^{\alpha+1}x^N|$
So that for $n \geq N$

$$|n^\alpha x^n| \leq \frac{1}{n}|N^{\alpha+1}x^N|$$

and thus $n^\alpha x^n \to 0$ as $n \to \infty$ by Q 2.

6) We have $\{\sin(\frac{\pi n}{2})\}_{n=1}^{\infty}$

There are many convergent subsequences
Take $n_k = (4k+1)$
Then $a_{n_k} = \sin\left(\frac{(4k+1)\pi}{2}\right) = \sin(2k\pi + \frac{\pi}{2})$
$= \sin(k\pi)\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})\cos(2k\pi)$
$= 1$

So subsequence is $\{1,1,1,..\}$
which is clearly convergent.

(7) The sequence is bounded above and below. The Bolzano-Weierstrass Theorem says that it has a convergent subsequence. We can take the lower bound to be b for the subsequence. Apply BW again to the subsequence and there is a sub-sub sequence which converges and the limit is $\geq b$

(8) Let n_k be any sequence of odd numbers $a_{n_k} = 1$.

Now $\frac{3^n + (-2)^n}{3^n - 2^n} = \frac{3^n(1+(\frac{-2}{3})^n)}{3^n(1-(\frac{2}{3})^n)} \to 1$ as $n \to \infty$.

So all subsequences converge to 1.

(9) If the sequence converges, all subsequences converge to the same limit and this is the limit of the sequence. $(2n)^{1/2n} \to l$, the limit of the sequence.

Now we can show $\{2^{\frac{1}{2n}} \to 1$ as $n \to \infty$.

Thus

$$n^{\frac{1}{2n}} \to l \text{ as } n \to \infty$$

Since

$$(2n)^{\frac{1}{2n}} = 2^{\frac{1}{2n}} n^{\frac{1}{2n}}$$

Hence

$$n^{1/n} = n^{1/2n} n^{1/2n} \to l.l = l^2 = l.$$

So either $l=1$ or $l=0$. We cannot have $l=0$, so $l=1$ and

$$n^{1/n} \to 1$$