Proof of the ratio test Recall that the test says that $\sum_{n=1}^{\infty} a_n$ converges if

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \quad \text{and } L<1. \text{ Here } a_n > 0 \text{ all } n.$$

We reduce the problem to the comparison test.

Suppose $L<1$. Pick $r \in \mathbb{R}$ such that $L<r<1$.

Now $\frac{a_{n+1}}{a_n} \to L$, so if $\varepsilon>0$, we can choose N large enough so that $\left|\frac{a_{n+1}}{a_n} - L\right| < \varepsilon$ for all $n \geq N$.

We pick $\varepsilon = r - L > 0$. $r-L<1$

So $n \geq N \Rightarrow \left|\frac{a_{n+1}}{a_n} - L\right| < r-L$.

This is the same as

$$-(r-L) < \frac{a_{n+1}}{a_n} - L < r-L.$$

Add L to get $\frac{a_{n+1}}{a_n} < r$

or $a_{n+1} < r a_n$, all $n \geq N$

This implies $a_n < r a_{n-1}$, but $a_{n-1} < r a_{n-2}$
$\therefore a_n < r^3 a_{n-2}$. We keep repeating this calculation and
In general $a_n < r^k a_{n-k}$.

Let $k = n-N$. Then

$$a_n \leq r^{n-N} a_N = \left(\frac{a_N}{r^N}\right) r^n, \quad n \geq N$$

Now the series $\sum_{n=1}^{\infty} \left(\frac{a_N}{r^N}\right) r^n$ converges. So by the comparison test, so does $\sum_{n=1}^{\infty} a_n$.

For $L>1$, the proof is similar but with $r>1$, $a_n > C_N r^n$ where $r>1$ implies $\sum_{n=1}^{\infty} C_N r^n$ diverges

For $L=1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and we still have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

So if $L=1$, convergence and divergence are both possible.

limsup and liminf If x_n is a sequence $\limsup_n x_n$ is the largest subsequential limit. Conversely the lim inf is the smallest subsequential limit

Example (1) $x_n = (-1)^n$

$$\{x_{2n}\} = \{(-1)^{2n}\} = 1, 1, 1 \ldots$$
$$\{x_{2n+1}\} = (-1)^{2n+1} = -1, -1, -1.$$

So $\limsup x_n = 1$

$\liminf x_n = -1$.

Proposition A sequence $\{x_n\}_{n=1}^{\infty}$ converges if and only if

$$\limsup x_n = \liminf x_n = (\lim x_n)$$

where both exist.

Theorem: (The nth root test) Let $\sum_{n=1}^{\infty} a_n$ be a series and suppose

$$\limsup |a_n|^{1/n} = L$$

Then the series converges if $L<1$, diverges if $L>1$ and gives no information if $L=1$.

Example Consider $\sum_{n=1}^{\infty}\left(\frac{2^n}{n!}\right)^n$.

Here $|a_n|^{\frac{1}{n}} = \left|\left(\frac{2^n}{n!}\right)^n\right|^{1/n} = \frac{2^n}{n!}$

So that $\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{2^n}{n!} = 0 < 1$.

$\left(\lim_{n\to\infty}\frac{2^n}{n!} = \limsup_{n\to\infty}\frac{2^n}{n!} = 0\right)$

$$\therefore \sum_{n=1}^{\infty}\left(\frac{2^n}{n!}\right)^n < \infty$$

Theorem (Cauchy Condensation Test)
Suppose that the sequence $\{a_n\}$ is positive and non increasing. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Moreover $\sum_{n=1}^{\infty} a_n \leq \sum_{n=0}^{\infty} 2^n a_{2^n} \leq 2\sum_{n=1}^{\infty} a_n$

Example $\sum_{n=1}^{\infty}\frac{1}{n^p}$, Then

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty}\frac{1}{2^{np-n}}$$

$$= \sum_{n=0}^{\infty} r^n, \quad r = \frac{1}{2^{p-1}}.$$

If $p>1$, $\frac{1}{2^{p-1}} < 1$ so $\sum_{n=1}^{\infty} r^n$ converges

If $p \leq 1$ $\frac{1}{2^{p-1}} \geq 1$ So $\sum_{n=1}^{\infty} r^n$ diverges

Hence $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$

and diverges for $p \leq 1$

Theorem (The alternating series test)
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms, with $a_{n+1} \le a_n$ and $a_n \to 0$.
Then
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ is convergent.}$$

Proof We know $a_n \to 0$. So if $\varepsilon > 0$ we can find N, s.t. $n \ge N \Rightarrow a_n < \varepsilon$
$a_n - a_{n+1} \ge 0$, since $\{a_n\}$ is monotone decreasing. We show
$$S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$
forms a Cauchy sequence. So it must converge. Notice

(*) $a_{m+1} - a_{m+2} + a_{m+3} - \dots + (-1)^{n+1} a_n \le a_{m+1}$.

Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$. Pick $n > m \ge N$

Then
$$|S_n - S_m| = \left| (a_1 - a_2 + a_3 - \dots + (-1)^{n+1} a_n) - (a_1 - a_2 + a_3 + \dots + (-1)^{m+1} a_m) \right|$$
$$= |a_{m+1} - a_{m+2} \dots + (-1)^{n+1} a_n|$$
$$\le |a_{m+1}| < \varepsilon$$
Hence $\{S_n\}$ is a Cauchy sequence $\therefore$ It converges.

Example Let $\{p_n\}_{n=1}^{\infty}$ be the nth prime.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{p_n}. \quad \frac{1}{p_n} \text{ is monotone decreasing}$$
So $\sum_{n=1}^{\infty} \frac{(-1)^n}{p_n}$ is convergent

(*) Note this follows because we are adding and subtracting terms that get smaller and smaller.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent since

$a_n = \frac{1}{n} \to 0$ monotonically

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2}$$

<u>Continuous Functions</u> A function is a mapping from one set X to another set Y. We write

$f : X \to Y$

We are interested in the case where $X, Y \subseteq \mathbb{R}$.

X is the <u>domain</u> of f and Y is the <u>range</u>

<u>Limits</u> We first define $\lim_{x \to a} f(x)$.

<u>Definition</u> We define limit points and limits of functions

(1) A point x is a limit point of a set $X \subseteq \mathbb{R}$ if there is a sequence $x_n \to x$ and $\{x_n\}_{n=1}^{\infty} \subset X$. If there is no such sequence, x is an isolated point

(2) Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and x_0 is a limit point of X. Then L is the limit of f as $x \to x_0$ if and only if, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in X$

$$|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Theorem Let $f, g : X \to \mathbb{R}$ be functions and c, a constants. Suppose that $\lim_{x \to x_0} f(x) = L$, $\lim_{x \to x_0} g(x) = M$

Then

(1) $\lim_{x \to x_0} c f(x) = cL$

(2) $\lim_{x \to x_0} (f(x) + g(x)) = L + M$

(3) $\lim_{x \to x_0} f(x) g(x) = LM$

(4) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided $g(x) \neq 0$ and $M \neq 0$.

Proof We do (3). Let $\varepsilon > 0$. Consider

$$|f(x)g(x) - LM|$$
$$= |f(x)g(x) - f(x)M + f(x)M - LM|$$
$$= |f(x)(g(x) - M) + M(f(x) - L)|$$
$$\leq |f(x)||g(x) - M| + |M||f(x) - L|$$

Choose δ such that $|x - x_0| < \delta_1$
$$\Rightarrow |f(x) - L| < \frac{\varepsilon}{2|M|}$$
for $x \in (x_0 - \delta, x_0 + \delta)$. Choose δ_2 such that $|g(x) - M| < \frac{\varepsilon}{2K}$. What is K going to be?

Note that f is near L if $x \in (x_0 - \delta, x_0 + \delta)$. In fact $L - \varepsilon < f(x) < \varepsilon + L$. Choose ε smaller than 1.

Then choose δ_2 such that $|x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2|L+1|}$

Then if $|x - x_0| < \delta$, $\delta = \min(\delta_1, \delta_2)$

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|$$
$$< |L+1| \frac{\varepsilon}{2|L+1|} + |M| \frac{\varepsilon}{2|M|} = \varepsilon$$

Continuity at $f: X \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $x \in X$, if for every sequence $\{x_n\} \subset X$, with $x_n \to x$ we have

$$\lim_{n \to \infty} f(x_n) = f(x)$$

Example $f(x) = x^2$. Let $x_n \to x$

Then $f(x_n) = x_n^2$

Let $\varepsilon > 0$. Then

$$|x_n^2 - x^2| = |(x_n - x)(x_n + x)|$$

$$= |x_n + x|\,|x_n - x|$$

x_n is convergent, hence it is bounded

Suppose that $|x_n| \le M > 0$ all n.

Now $|x_n + x| \le |x_n| + |x|$

$$\le M + |x|$$

So

$$|x_n^2 - x^2| \le (M + |x|)|x_n - x|$$

Choose N st. $n \ge N \Rightarrow |x_n - x| < \dfrac{\varepsilon}{M + |x|}$

Then $n \ge N \Rightarrow |x_n^2 - x^2| < (M + |x|)\dfrac{\varepsilon}{M + |x|}$

$$= \varepsilon$$

So $x_n^2 \to x^2$.

Thus f is continuous at x, for any x.