Definition Let $f: X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$. We say that $\lim_{x \to a^+} f(x) = L$,

if for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that for $a < x < a+\delta$ $|f(x) - L| < \varepsilon$.

We say that $\lim_{x \to a^-} f(x) = L$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $a - \delta < x < a$ $|f(x) - L| < \varepsilon$

Proposition Let $f: X \to \mathbb{R}$, where $X \subseteq \mathbb{R}$, $a \in X$. Then $\lim_{x \to a} f(x) = L$

if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Proof This follows from the definitions.

if $a - \delta < x < a$ and $a < x < a + \delta$

then $|x - a| < \delta$ and $|f(x) - L| < \varepsilon$

So $\lim_{x \to a} f(x) = L$

We can also consider the limit at infinity.

Definition Let $f: \mathbb{R} \to \mathbb{R}$. Then $\lim_{x \to \infty} f(x) = L$

if for every $\varepsilon > 0$ there is an $M > 0$, st. $x \geq M \Rightarrow |f(x) - L| < \varepsilon$.

Similarly $\lim_{x \to -\infty} f(x) = L$ if for every $\varepsilon > 0$

there is an $M < \infty$ such that if $M \leq x$ then $|f(x) - L| < \varepsilon$

The composition of functions If f, g are continuous functions with $g: X \to Y$ and $f: Y \to \mathbb{R}$, $(f \circ g)(x) = f(g(x))$ and $f \circ g : X \to \mathbb{R}$.

Example (1) $f(x) = x^2$ $g(x) = x^3$

$$(f \circ g)(x) = f(g(x)) = (x^3)^2 = x^6$$

(2) $f(x) = \sin x$ $g(x) = \sqrt{x^2+1}$

$$(g \circ f)(x) \overset{\circ}{=} g(f(x)) = \sqrt{1+\sin^2 x}$$

The $\varepsilon - \delta$ definition of continuity

Def 1.

Let $f: X \to \mathbb{R}$ $X \subseteq \mathbb{R}$. f is continuous at $x \in X$, if given any sequence $\{x_n\}$ with $x_n \to x$, $f(x_n) \to f(x)$.

There is another definition

Definition 2 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ and let $x \in X$. We say that f is continuous at x if given $\varepsilon > 0$, we can find $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

In general δ depends on x. Sometimes we write δ_x.

Theorem Definition 1 and definition 2 are equivalent

Proof Suppose f satisfies Defn 2. Let $\delta, \varepsilon > 0$. pick a sequence $\{x_n\}$ with $x_n \to x$. Then we can find an $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |x - x_n| < \delta$. However this means $|f(x) - f(x_n)| < \varepsilon$. Thus $f(x_n) \to f(x)$

So $x_n \to x \Rightarrow f(x_n) \to f(x)$

Def 1 $\Rightarrow$ Def 2 is an exercise

Third definition First

$A \subseteq \mathbb{R}$ is open if for any $a \in A$ we can find $\varepsilon > 0$ such that $B_\varepsilon(a) = \{x \in \mathbb{R}, |x - a| < \varepsilon\}$ is contained in A

$X = (0,1)$. let $a = 1/2$ $\left(\frac{1}{4}, \frac{3}{4}\right) \subset X$

A is closed if its compliment is open
$A^c = \{x \in \mathbb{R}, x \notin A\}$

Definition $f: X \Rightarrow \mathbb{R}$ is continuous if
$f^{-1}(A) = \{x \in X : f(x) \in A\}$ is open whenever A is open. This is equivalent to Defn 1 and Defn 2

$f(x) = x^2$, $A = (0,1)$ $X = \mathbb{R}$
Then $f^{-1}(A) = (-1,1)$ $(x \in (-1,1) \Rightarrow f(x) \in A)$

Continuous functions have the basic properties you would expect
If f, g are continuous at x
$f+g$ is continuous fg is continuous
$\frac{f}{g}$ is continuous, if $g \neq 0$.

For 2 $|f(x)g(x) - f(y)g(y)|$
$= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)|$

$\leq |f(x) - f(y)||g(x)| + |f(y)||g(x) - g(y)|$
etc. It is the same idea we have seen before.

Theorem Let $g: X \subseteq \mathbb{R} \to Y$, $f: Y \to Z \subseteq \mathbb{R}$. If g, f are continuous, then $f \circ g$ is continuous.

Proof
$(f \circ g)(x) = f(g(x)) : X \to Z$
what is $(f \circ g)^{-1}(A)$? We have
$(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A))$
(A is open) $= g^{-1}(B)$
where $f^{-1}(A) = B$ which is open since f is cts.
But $g^{-1}(B)$ is open by continuity.
$\therefore (f \circ g)^{-1}(A)$ is open, So $f \circ g$ is continuous

Example $f(x) = \sin x$, $g(x) = x^2$
$(f \circ g)(x) = \sin(x^2)$ which is cont.
$(g \circ f)(x) = \sin^2 x$ is continuous

We can also define left and right continuity. See notes

Uniform Continuity. A function $f: X \to \mathbb{R}$ is uniformly continuous if given $\varepsilon > 0$ we can find $\delta > 0$ st. whenever $|x-y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$.

Theorem
If $f: [a,b] \to \mathbb{R}$ is continuous it is also uniformly continuous. Here a, b are finite.

To prove this, we use sequential uniform continuity. A function is sequentially uniformly continuous if given $x_n\, y_n$
$y_n - x_n \to 0 \Rightarrow f(y_n) - f(x_n) \to 0$
This is equivalent to uniform continuity.

Proof of Theorem
Let f be continuous, but not uniformly continuous. It is not sequentially u.c.
Choose $r > 0$ such that for every $\delta > 0$ there exists $x, y \in [a,b]$ with $|x-y| < \delta$ and $|f(x) - f(y)| > r$
For each $N \in \mathbb{N}$ choose $x_n, y_n \in [a,b]$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq r$
By Bol.-Wei $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}$. Suppose $x_{n_k} \to x$
$y_{n_k} \to ?$ $\{x_{n_k} - y_{n_k}\}$ is a subsequence of $\{x_n - y_n\}$. So $\{x_{n_k} - y_{n_k}\}$ is convergent

and $x_{n_k} - y_{n_k} \to 0$ since $x_n - y_n \to 0$
So

$$y_{n_k} = \underbrace{x_{n_k}}_{\to x} - \underbrace{(x_{n_k} - y_{n_k})}_{\to 0}$$

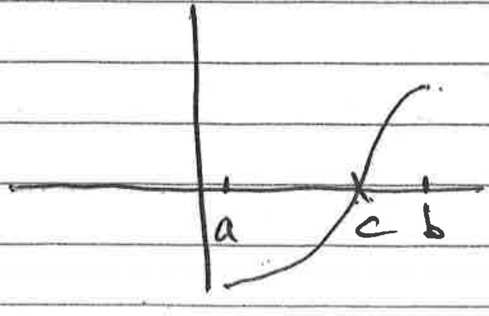
$\therefore\ y_{n_k} \to x.$

f is continuous. So $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$. So $f(x_{n_k}) - f(y_{n_k}) \to 0$. But we assumed $|f(x_{n_k}) - f(y_{n_k})| \geq r > 0$. This is a contradiction.

Definition A function $f: X \to \mathbb{R}$ is Lipschitz continuous if there is a constant M such that for every $x, y \in X$

$$|f(x) - f(y)| \leq M|x - y|.$$

Maxima and Minima A continuous function on a closed and bounded interval $[a,b]$ is bounded. Moreover it attains its maximum and minimum values on $[a,b]$.
Proof See notes
Theorem Suppose $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $f(a)f(b) < 0$. Then there is a $c \in [a,b]$ s.t. $f(c) = 0$.
Proof



$f(b)f(a) < 0$

Suppose $f(a) < 0$, $f(b) > 0$

Let $A = \{x \in [a,b] : f(x) < 0\}$. Then $a \in A$, so A is nonempty and bounded above. It therefore has a least upper bound which we call c. Choose x_n such that $c - \frac{1}{n} < x_n \le c$. Then $f(x_n) < 0$

By continuity $\lim_{n\to\infty} f(x_n) = f(c) \le 0$

Take $y_n = c + \frac{(b-c)}{n}$. Then $y_n \to c$

$f(y_n) > 0$ all n.

$\lim_{n\to\infty} f(y_n) = \underbrace{f(c) \ge 0}_{*}$ by continuity

$\therefore$ $f(c) \le 0$ and $f(c) \ge 0$

So $f(c) = 0$.

(*) since $f(y_n) > 0$ all n $\lim_{n\to\infty} f(y_n) \ge 0$.

<u>Corollary</u> Let f be continuous on $[a,b]$. Suppose that $f(a) \ne f(b)$ and that M lies between $f(a)$ and $f(b)$. Then there is a $c \in [a,b]$ st $f(c) = M$.

<u>Proof</u> Apply previous theorem to

$g(x) = f(x) - M$

Then g satisfies assumptions of previous result so $\exists$ c st $g(c) = 0$

$\therefore$ $f(c) - M = 0$