Corollary If f is continuous on $[a,b]$ and differentiable on (a,b), and $f'(x)=0$ all $x\in(a,b)$ then f is constant on $[a,b]$

Proof For any $x,y\in(a,b)$, $f(x)-f(y)=f'(c)(x-y)=0$

So $f(x)=f(y)$ all $x,y\in(a,b)$

By continuity it is also constant on $[a,b]$

As an application we prove a uniqueness result.

Proposition The equation $y'=ky$, $y(0)=y_0$ has a unique solution

Proof Let $y(x)=y_0 e^{kx}$. This is a solution (Just check it). Now let f be another solution and define

$$h(x)=f(x)e^{-kx}$$

Then
$$h'(x)=f'(x)e^{-kx}-kf(x)e^{-kx}$$
$$=kf(x)e^{-kx}-kf(x)e^{-kx}$$
$$=0$$

So h is a constant

Let $h(x)=C$.

ie. $f(x)e^{-kx}=C$

$$\therefore\ f(x)=Ce^{kx}$$

and $f(0)=Ce^{0}=k$.

So $f(x)=y(x)$. Therefore there is only on solution.

The Generalised Mean Value Theorem and L'Hôpital's rule

Suppose that f and g are continuous on $[a,b]$, which are differentiable on (a,b), and $g'(x)\neq 0$, for all $x\in(a,b)$ Then there exists $c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

Proof First the MVT says that $\exists\ c \in (a,b)$ with $\frac{g(b)-g(a)}{b-a} = g'(c)$

Since $g'(c) \neq 0$ $(b \neq c)$ $g(b) \neq g(a)$

So $g(b) - g(a) \neq 0$

The trick is to find the right function to apply Rolle's Theorem to. We take

$$h(x) = f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)].$$

Note h' exists

$$h(a) = f(a)[g(b)-g(a)] - g(a)[f(b)-f(a)]$$

$$= f(a)g(b) - f(b)g(a) = h(b)$$

So by Rolle's Theorem $\exists\ c$ st. $h'(c)=0$

or

$$f'(c)[g(b)-g(a)] - g'(c)[f(b)-f(a)] = 0$$

or $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

L'Hôpital's Rule. Suppose that f and g are differentiable on (a,b) and $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose further that $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$.

Then $$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$$

provided the limit exists on the right.

Proof Suppose $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$.

Given $\varepsilon > 0$ we can find $\delta > 0$ such that if $c \in (a, a+\delta)$ $\left| \frac{f'(c)}{g'(c)} - L \right| < \varepsilon$

By the Generalised Mean Value Theorem
if $x \in (a, a+\delta)$

$$\left|\frac{f'(\alpha)}{g'(\alpha)} - L\right| = \left|\frac{f(x)-f(a)}{g(x)-g(a)} - L\right| < \varepsilon$$

$$= \left|\frac{f(x)}{g(x)} - L\right| < \varepsilon \qquad \begin{pmatrix} f(a) \\ = g(a) = 0 \end{pmatrix}$$

$$\therefore \lim_{x \to a^+} \frac{f(x)}{g(x)} = L = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

This is true for $\lim_{x \to a} \frac{f(x)}{g(x)}$ when we have $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Example (1) $\lim_{x \to 0} \frac{\sin x}{x}$ This has form $\frac{0}{0}$

$$\frac{\frac{d}{dx}\sin x}{\frac{d}{dx}x} = \frac{\cos x}{1}$$

$\lim_{x \to 0} \frac{\cos x}{1} = 1.$ By L'Hôpital's rule

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}\sin x}{\frac{d}{dx}x} = 1$$

(2)
$\lim_{x \to \infty} \frac{x^2+4x+3}{3x^2+2x+9}$ has the form $\frac{\infty}{\infty}$

$$\frac{\frac{d}{dx}(x^2+4x+3)}{\frac{d}{dx}(3x^2+2x+9)} = \frac{2x+4}{6x+2}$$

$$\lim_{x \to \infty} \frac{2x+4}{6x+2} \quad \left(\frac{\infty}{\infty}\right)$$

$$\frac{\frac{d}{dx}(2x+4)}{d/dx(6x+2)} = \frac{2}{6} = \frac{1}{3}$$

$$\lim_{x\to\infty} \frac{x^2+4x+3}{3x^2+2x+9} = \lim_{x\to\infty} \frac{2x+4}{6x+2} = \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}$$

Inverse Functions A function $f: X \to Y$ is one-to one if for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$. If $f: X \to Y$ is one to one there is an inverse function

$$f^{-1}: Y \to X$$

and $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

When does f^{-1} exists. If f is continuous and $f'(x) > 0$ for all $x \in X$. Then f^{-1} exists for $f(x) \in Y$.

Example $f(x) = e^x$, $x > 0$. The inverse is $\ln x$.

$$e^{\ln x} = x. \quad \ln e^x = x$$

For $f(x) = x^2$, $x > 0$

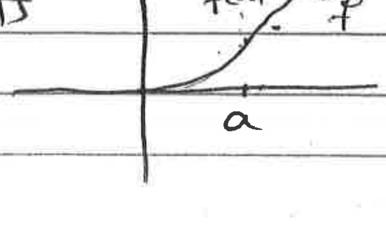
$$f^{-1}(x) = \sqrt{x}, \quad \sqrt{x^2} = x$$

$$(\sqrt{x})^2 = x \quad \text{for } x > 0$$

Theorem (The Inverse Function Theorem) Suppose that f is differentiable and one to one on an open interval I. If $f'(a) \neq 0$ $a \in I$, then f^{-1} exists and is differentiable at $f(a)$ and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Proof f' is nonzero, so f is increasing or decreasing on I, so f^{-1} exists



Take $y = f(x)$

$$\lim_{y\to f(a)} \frac{f^{-1}(y)-f^{-1}(f(a))}{y-f(a)} = \lim_{f(x)\to f(a)} \frac{f^{-1}(f(x))-f^{-1}(f(a))}{f(x)-f(a)}$$

$$= \lim_{x\to a} \frac{x-a}{f(x)-f(a)}$$

$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)^{-1} = \frac{1}{f'(a)}$$

This is often written $\frac{dx}{dy} = \frac{1}{dy/dx}$

<u>Example</u> $f(x)=x^2$, $a=2$ $(x>0)$

$f'(x)=2x$.

$\therefore f'(2)=4$

$\therefore \frac{d}{dx}(f^{-1}(f(2))) = \frac{1}{f'(2)} = \frac{1}{4}$.

To check

$f^{-1}(x) = x^{1/2}$. $\frac{d}{dx}(x^{1/2}) = \frac{1}{2\sqrt{x}}$

$f(2)=4$.

$\therefore \frac{d}{dx} f^{-1}(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

<u>Power Series and Taylor Expansions</u>

A power series about a point x_0 is an expression

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

Many functions can be written this way

eg. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

By the ratio test a power series will converge if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = L < 1$$

or for all x such that

$$|x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

the values of x for which this holds is the <u>interval of convergence</u>.

If the series converges for $|x-x_0|<R$, then it converges for all

$$x \in (x_0-R,\ x_0+R)$$

R is the <u>radius of convergence</u>.

<u>Example</u> The series $1+x+x^2+x^3+\ldots = \frac{1}{1-x}$

This is the Geometric series

Geometric series. $a_n = x^n$

We apply the ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x|\lim_{n\to\infty} 1$$

$$= |x| < 1$$

for $x \in (-1,1)$

So Radius of convergence is 1

<u>Theorem</u> Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then the series converges absolutely for $|x|<R$ and diverges for $|x|>R$

Theorem Let $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R. Then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has radius of convergence R.

If $f : (-R, R) \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Then f is differentiable on $(-R, R)$

$$\text{and } f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Definition A function is *smooth* if it is infinitely differentiable on an interval I. We say $f \in C^{\infty}(I)$

[If $f \in C^n(I)$ it is n times continuously differentiable.]

If f is a smooth function around $a \in I$, the Taylor series of f about $a \in I$ is

$$\cancel{Tf}\ T_f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + f^{(iv)}(a)\frac{(x-a)^4}{4!} + \dots$$

If f equals its Taylor series it is *analytic*.

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$\therefore$ $f(b) = f(a) + f'(c)(b-a)$ — Mean Value Theorem. Taylor's Theorem extends this

Example $f(x) = e^x$, $a = 0$

$f'(x) = e^x$. $\therefore f'(0) = 1$.

$f''(x) = e^x$. $\therefore f''(0) = 1$

In fact $f^{(n)}(0) = 1$

$$\therefore T_f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let $g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$$g(.3) = 1 + .3 + \frac{(.3)^2}{2} + \frac{(.3)^3}{6} = 1.3495$$

$e^{0.3} = 1.3498$. So we get a very good approximation

Example $f(x) = \sin x$, $a = 0$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(iv)}(x) = \sin x$

$$T_f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \frac{d}{dx} \sin x = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

We often define functions as Taylor series. (See differential equations)

$$\sin(0.3) \approx 0.3 - \frac{(0.3)^3}{3!}$$

<u>Taylor's Theorem.</u>