Integration. We start by defining a partition P. We take an interval $[a,b]$ and introduce a set of points $\{x_0, \ldots, x_n\}$ where

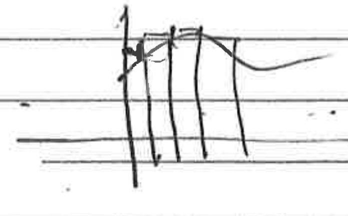
$$a = x_0 < x_1 < \ldots < x_n = b$$

Then the interval $[a,b]$ is the union of the subintervals, $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$.
The length of the partition is

$$|P| = \max_{i=1,\ldots,n} \{ |x_i - x_{i-1}| \}$$

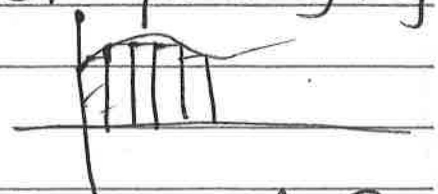
We take a function f on $[a,b]$ and create a set of Riemann sums (Named after Georg Friedrich Bernhard Riemann 1826-1866, one of the greatest mathematicians in history).
Assume that f is bounded.
Let $\bar{x}_i \in [x_{i-1}, x_i]$ be the point where f has its maximum. $M_i = f(\bar{x}_i)$
Let $\underline{x}_i \in [x_{i-1}, x_i]$ be the point where f has its minimum. $m_i = f(\underline{x}_i)$
Then

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$



is the upper Riemann sum corresponding to P.

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$



is the lower Riemann sum corresponding to P.
Different partitions will give different upper and lower sums. We define

$$\overline{\int_a^b} f = \inf \{ U(f,P), P \text{ a partition of } [a,b] \}$$

– upper integral.

$\underline{\int_a^b} f = \sup\{ L(f,P) : P \text{ a partition of } [a,b] \} = B$

– lower integral

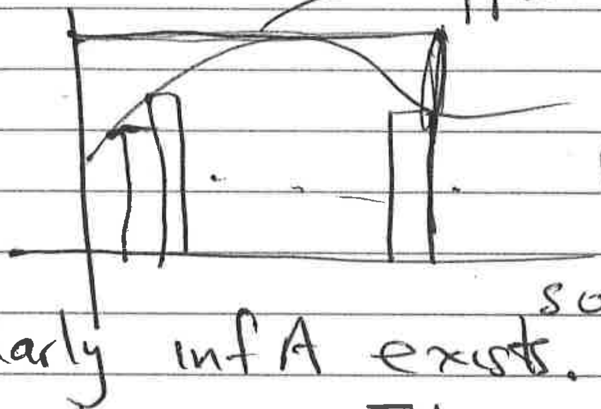
If

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

Then f is Riemann integrable and

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f = \underline{\int_a^b}.$$

Note



Area in the box is an upper bound for $\{L(f,P), P \text{ a partition}\}$

Thus B is non-empty, bounded above so $\sup B$ exists

Similarly $\inf A$ exists.

So $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ exist. When are they equal?

Some elementary facts follow.

Proposition The Riemann integral has the following properties

(1) If f is a constant c, $\int_a^b c\,dx = c(b-a)$

(2) $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

Exercise (2) is just triangle inequality

Theorem (Riemann's Criterion) Let f be a bounded function on the closed interval $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if, given any $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon$$

Theorem

Every continuous function on a closed and bounded interval $[a,b]$ is Riemann integrable.

Proof f is continuous, so it is bounded on $[a,b]$. Let $\varepsilon > 0$. f is continuous so it is uniformly continuous. So we can choose $\delta > 0$ such that if $x, y \in [a,b]$, with $|x-y| < \delta$ then $|f(x) - f(y)| < \dfrac{\varepsilon}{b-a}$.

Now choose $N \in \mathbb{N}$, $N > \dfrac{(b-a)}{\delta}$.

For each $i = 0, 1, \ldots, N$, let $x_i = a + \dfrac{(b-a)i}{N}$.

Then $P = \{x_0, \ldots, x_N\}$ is a partition of $[a,b]$ and

$$|x_i - x_{i-1}| < \delta.$$

By continuity f attains its maximum and minimum on $[x_{i-1}, x_i]$. Now let

$$f(c_i) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$
$$f(d_i) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

Now $|d_i - c_i| < \delta$ and $f(d_i) \geq f(c_i)$.

By uniform continuity

$$f(d_i) - f(c_i) < \frac{\varepsilon}{b-a}.$$

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} (f(d_i) - f(c_i))(x_i - x_{i-1}) \\
&< \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) \\
&= \frac{\varepsilon}{b-a}(x_1 - x_0 + x_2 - x_1 \cdots + x_n) \\
&= \frac{\varepsilon}{b-a}(b-a) = \varepsilon
\end{aligned}
$$

(This is property 1 of the prop on p 60)

So f is integrable

How do we calculate integrals?
One idea. Use Riemann sums

Example $f(x) = x^2$ on $[0,1]$. Since f is increasing take $P = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$

Recall
$$\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$m_i = \inf\{x^2 : x \in [\tfrac{i-1}{n}, \tfrac{i}{n}]\}$$
$$= \frac{(i-1)^2}{n^2}$$
$$M_i = \sup\{x^2 : x \in [\tfrac{i-1}{n}, \tfrac{i}{n}]\}$$
$$= \frac{i^2}{n^2}$$
$$L(f,P) = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2}(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} \frac{(i-1)^2}{n^3}$$

Also
$$U(f,P) = \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$
$$\int_a^b f(x)\,dx = \overline{\int_a^b} f \qquad \text{(because } f \text{ is cont.)}$$
$$= \inf_{n \geq 1} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$$

Since
$$\frac{n(n+1)(2n+1)}{6n^3} = \frac{2n^2+n+2n+1}{6n^2}$$
$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \to \frac{1}{3} \text{ as } n \to \infty$$

$\left[\text{or } \int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}\right]$

This is useful in simple cases, but not if $f(x) = x^n$, n large.

There is a better way!!

The Fundamental Theorem of Calculus

If f is a continuous function on $[a,b]$ then for all $x \in [a,b]$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Proof Define

$$F(x) = \int_a^x f(t)\,dt$$

f is continuous, hence it is bounded on $[a,b]$.

So $\exists$ $M > 0$, such that $|f(t)| \leq M$ all $t \in [a,b]$

Then

$$|F(x) - F(y)| = \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right|$$

$$= \left| \int_y^x f(t)\,dt \right|$$

$$\leq \int_y^x |f(t)|\,dt$$

$$\leq \int_y^x M\,dt \leq M|x-y|.$$

So F is Lipschitz continuous. It is also uniformly continuous and continuous

Now

$$\frac{F(x) - F(y)}{x-y} - f(y) = \frac{1}{x-y}\left(F(x) - F(y) - (x-y)f(y)\right)$$

$$= \frac{1}{x-y}\int_y^x \left(f(t) - f(y)\right)dt$$

Note

$$\int_y^x f(y)\,dt = (x-y)f(y)$$

By uniform continuity of f, given $\varepsilon > 0$ we may choose $\delta > 0 \Rightarrow |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Choose such values of δ and ε

$$\left|\frac{F(x)-F(y)}{x-y}-f(y)\right| \leq \frac{1}{|x-y|}\int_y^x |f(t)-f(y)|\,dt$$

$$< \frac{\varepsilon}{|x-y|}\int_y^x dt$$

$$= \frac{\varepsilon}{|x-y|}(x-y) = \varepsilon$$

since $x>y$. Thus F' exists and $F'=f$

Corollary (The Fundamental Theorem of Calculus II. - The return of the fundamental theorem)

Let f be Riemann integrable function on $[a,b]$. Then, if $F'=f$ on (a,b) then

$$\int_a^b f(x)\,dx = F(b)-F(a) = \left[F(x)\right]_a^b.$$

Proof Suppose that $G(x)=\int_a^x f(t)\,dt$ and $F'=f$.

$\therefore$ $G-F$ is a constant, since $G'=F'$,

Hence $G(b)-F(b) = G(a)-F(a)$

But

$$G(a) = \int_a^a f(t)\,dt = 0$$

So $$G(b) = \int_a^b f(t)\,dt = F(b)+G(a)-F(a)$$

$$= F(b)-F(a).$$

The integral has many useful properties

Mean Value Theorem for Integrals

Suppose that f and g are continuous on $[a,b]$ and $g(x)\geq 0$ for all $x\in[a,b]$ then there exists $c\in[a,b]$ such that

$$\int_a^b f(x)g(x) = f(c)\int_a^b g(x)dx.$$

Proof Notes

Integration by parts

We know $\frac{d}{dx}(f(x)g(x)) = f'g+g'f$

So $f'g = (fg)' - g'f$

Hence $$\int_a^b f'(x)g(x)dx = \int_a^b (f(x)g(x))'dx - \int_a^b g'(x)f(x)dx$$
$$= \left[f(x)g(x)\right]_a^b - \int_a^b g'(x)f(x)dx$$

Integration by substitution. This is most important.

$$\int_a^b f'(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

where we set $u = g(x)$

Improper Riemann Integrals

Let $f:[a,b]\to\mathbb{R}$ be continuous, but possess a discontinuity at a

We can define $$\int_a^b f(x)dx = \lim_{X\to a}\int_X^b f(x)dx$$

More generally if f is defined on $\mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{T\to\infty}\int_{-T}^{0} f(x)\,dx + \lim_{R\to\infty}\int_{0}^{R} f(x)\,dx$$

These are improper Riemann integrals.
Provided they exist

Example Calculate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$$\int_0^1 x^{-1/2}\,dx = \lim_{X\to 0}\int_X^1 x^{-1/2}\,dx$$

$$= \lim_{X\to 0} 2\sqrt{x}\Big]_X^1$$

$$= \lim_{X\to 0} 2\sqrt{1} - 2\sqrt{X}$$

$$= 2$$

Example $$\int_0^\infty \frac{dx}{x^2+1} = \lim_{R\to\infty}\int_0^R \frac{dx}{1+x^2}$$

$$= \lim_{R\to\infty} \tan^{-1}x\Big]_0^R$$

$$= \lim_{R\to\infty}\left(\tan^{-1}R - \tan^{-1}0\right)$$

$$= \lim_{R\to\infty}\tan^{-1}R - 0 = \frac{\pi}{2}$$

There are numerous techiques for computing these kinds of integrals