RA Tutorial 9 Solutions

(1) $\sum_{k=1}^{n} k^4 = = An^5 + Bn^4 + Cn^3 + Dn^2 + En + F$

Take $n = 1, 2, 3, 4, 5$ to get a set of equations for $A, B, \ldots F$.

You get, $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$.

These formulae get complicated quickly, so doing integration with them becomes prohibitive for large powers.

(2) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (From previous work)

To evaluate the integral $\int_0^1 (x^4 + 3x^2 + 2x)\,dx$

we partition $[0,1]$ into $[0,\frac{1}{n}], [\frac{1}{n}, \frac{2}{n}) \ldots, [\frac{n-1}{n}, 1]$.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Set up the Riemann sums $\left(\sum f(x_i)(x_i - x_{i-1})\right)$

$$\sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 \left(\frac{i}{n} - \frac{(i-1)}{n}\right) + 3\sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{i}{n} - \frac{(i-1)}{n}\right)$$

$$+ 2\sum_{i=1}^{n} \left(\frac{i}{n}\right)\left(\frac{i}{n} - \frac{(i-1)}{n}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4 + \frac{3}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 + \frac{2}{n}\sum_{i=1}^{n}\frac{i}{n}$$

$$= \frac{1}{n^5}\sum_{i=1}^{n} i^4 + \frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{1}{n^5}\,\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + \frac{3}{n^3}\,\frac{(n(n+1)(2n+1))}{6}$$

$$+ 2 \cdot \frac{1}{n^2}\,\frac{n(n+1)}{2}$$

Now take $\lim_{n\to\infty}$

$$= \lim_{n\to\infty}\left(\frac{6n^5}{30n^5} + \text{lower terms}\right)$$

$$+ \lim_{n\to\infty} \frac{3}{n^3}\left(\frac{2n^3}{6} + \frac{n^2}{2} + \frac{n}{6}\right) + 2\lim_{n\to\infty} \frac{1}{n^2}\frac{(n^2+n)}{2}$$

$$= \frac{1}{5} + 1 + 1 = 2\tfrac{1}{5}.$$

Notice this is a laborious way of doing integration. See next question

3) $\int_a^b x^n dx = \left[\frac{x^n}{n+1}\right]_a^b$ since $\frac{d}{dx}\frac{x^n}{n+1}$

Notice how much easier this is than using Riemann sums

(4) Let $f(x) = x^\alpha$. Partition $[0,1]$ as

$$\left[0,\tfrac{1}{n}\right), \left[\tfrac{1}{n},\tfrac{2}{n}\right), \ldots \left[\tfrac{n-1}{n}, 1\right]$$

Now $\int_0^1 x^\alpha dx = \frac{1}{1+\alpha}$.

The Riemann sums must converge to this

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

Take x_i to be left end point of partition intervals.

$$= \sum_{i=1}^{n}\left(\frac{i}{n}\right)^\alpha\left(\frac{i}{n} - \frac{(i-1)}{n}\right)$$

$$= \frac{1}{n^{\alpha+1}}\sum_{i=1}^{n} i^\alpha$$

$$= \frac{1}{n^{\alpha+1}}(1^\alpha + 2^\alpha + \cdots + n^\alpha) \to \int_0^1 x^\alpha dx = \frac{1}{1+\alpha}$$

as $n \to \infty$

(5) $\int_a^b (tf(x)+g(x))^2 \, dx \geq 0$.

Now $\int_a^b (tf(x)+g(x))^2 dx = t^2\int_a^b (f(x))^2 dx + 2t\int_a^b f(x)g(x)dx + \int_a^b g(x)^2 dx \geq 0$

Now

$$At^2 + Bt + C \geq 0 \Rightarrow B^2 \leq 4AC$$

Let $A = \int_a^b (f(x))^2 dx$

$$B = 2\int_a^b f(x)g(x)dx$$

$$C = \int_a^b (g(x))^2 dx.$$

So

$$4\left(\int_a^b f(x)g(x)dx\right)^2 \leq 4\int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx$$

which gives the result

(6) $\int_a^b g(x)dx \geq 0$ if $g(x) \geq 0$. By continuity if $g>0$, then it is >0 on a finite interval. Suppose $g(x) > a > 0$, on $(\varepsilon, \delta) \subset (a,b)$ $\delta > 0$.
Then

$$\int_a^b g(x)\,dx \geq \int_\varepsilon^\delta g(x)dx > a\int_\varepsilon^\delta dx$$

$$= a(\delta - \varepsilon) > 0.$$

So if g is nonzero and non negative the integral is nonzero. Since $\int g = 0$ and g is continuous $g = 0$

(7) Use integration by parts

$$\int_a^b x f''(x)dx = [xf'(x)]_a^b - \int_a^b f'(x)dx$$
$$= bf'(b) - af'(a) - [f(x)]_a^b$$
$$= bf'(b) - f(b) - (af'(a) - f(a)).$$

(8) $F(x) = \int_1^x f(t)\,dt \le (f(x))^2, \; x \ge 1$

$$G(x) = \int_1^x \frac{F'(t)}{\sqrt{F(t)}}\,dt$$

Now $F'(t) = f(t) \ge (F(t))^{1/2}$

Now $x-1 = \int_1^x 1\,dt \le \int_1^x \frac{F'(t)}{\sqrt{F(t)}}\,dt$

Since if $f(t) \le g(t)$ on $[a,b]$ $\int_a^b f(t)\,dt \le \int_a^b g(t)\,dt$

$$\int_1^x \frac{F'(t)}{\sqrt{F(t)}}\,dt = \left[2(F(t))^{1/2}\right]_1^x \quad (F(1)=0)$$

$$= 2(F(x))^{1/2} \le 2f(x)$$

(9) $\lim\limits_{R\to\infty} \int_0^R \frac{dx}{(1+x^2)(4+x^2)} = \frac{1}{3}\lim\limits_{R\to\infty} \int_0^R \left[\frac{1}{x^2+1} - \frac{1}{x^2+4}\right]dx$

$$= \frac{1}{3}\lim_{R\to\infty}\left[\tan^{-1}x - \frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^R$$

$$= \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12} < \infty$$

(10) $\lim\limits_{\varepsilon\to 0} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} = \lim\limits_{\varepsilon\to 0}\left[2\sqrt{x}\right]_\varepsilon^1$

$$= 2 - \lim_{\varepsilon\to 0} 2\sqrt{\varepsilon} = 2$$