Exam Instructions

37161 Probability and Random Variables

Assessment Task 3 – Take Home Examination

Spring 2023

Instructions

This examination is made available online at 12:30pm on Friday 24 November 2023

Your completed answer file is due at 3:30pm on Friday 24 November 2023 and must be submitted online via the Final Exam link in the Assignments folder on Canvas.

There are 5 questions. Your answer to each question attempted should commence on a new page and be appropriately numbered.

The examination is worth 50% of the marks available in this subject. The contribution each question makes to the total examination mark is indicated in marks.

This examination is an open book examination.

This examination is expected to take approximately 2 hours of working time. You are advised to allocate your time accordingly. Your answer file may be submitted at any time before the due time. Please allow time to complete the submission process.

Please submit your file in PDF format where possible. Please name your file as follows:

EXAM_subject number_student number e.g. EXAM_54000_12345678

Important Notice – Exam Conditions and Academic Integrity

In attempting this examination and submitting an answer file, candidates are undertaking that the work they submit is a result of their own unaided efforts and that they have not discussed the questions or possible answers with other persons during the examination period. Use of Generative Artificial Intelligence (Gen-AI) is prohibited. Candidates who are found to have participated in any form of cooperation or collusion or any activity which could amount to academic misconduct in the answering of this examination will have their marks withdrawn and disciplinary action will be initiated on a complaint from the Examiner.

Exam answers must be submitted through Canvas.. Vivas or other invigilated tasks may be used to verify student achievement of learning outcomes to ensure they have completed the work on their own and to assess their knowledge of the answers they have submitted.

Students must not post any requests for clarification on the Discussion Boards on Blackboard, Canvas or Microsoft Teams. Any requests for clarification should be directed by email to Stephen Woodcock on stephen.woodcock@uts.edu.au. Where clarification is required it will be broadcast by email to all students in the exam group. Stephen Woodcock will be available on email during the first hour of the examination time)

Question 1 ((2+2+2+2+2+2+2) + (3+3) = 20 marks)

a) A company employs 40 men and 60 women. At the company's Christmas party, five prizes are given out to employees such that each prize is awarded at random, with each of the 100 employees equally likely to receive each of the prizes.

Assume that the allocation of each prize is independent of that of all other prizes so that it is possible for a single employee to be randomly selected to win more than one prize.

Calculate the probability that:

i) Exactly two of the five prizes are won by women;

ii) The first and second prize drawn are both won by men;

iii) All five prizes go to employees of the same sex (i.e. all are won by men or all are won by women).

iv) Show that the probability that all five prizes are won by different employees $\approx 90.3\%$.

Assume now that after each prize is won, the winning employee's name is removed from consideration for future prizes. That is, the each employee wins the first prize awarded with probability $\frac{1}{100}$ and the second prize is won by each of the employees who did not with the first prize with probability $\frac{1}{99}$ etc.

Calculate the probability that:

v) The first and second prize drawn are both won by men;

vi) All five prizes go to employees of the same sex (i.e. all are won by men or all are won by women).

vii) All five prizes are won by different employees

Over...

Question 1 (Continued)

b) A business sells electronics equipment. 70% of its equipment sales are of new goods, 10% are of second-hand goods and 20% are of factory refurbished goods.

The business surveys customer satisfaction and finds that 95% of new equipment, 80% of second-hand equipment and 90% of factory refurbished equipment purchased is deemed satisfactory. (All purchases are classified as either satisfactory or unsatisfactory.)

i) Show that overall 7.5% of equipment purchased is deemed unsatisfactory.

ii) A randomly selected customer deemed the equipment he purchased to be unsatisfactory.
Calculate the probability that his equipment was purchased new.

Over...

Question 2 ((9) + (2 + 2) + (1+1+1+1+1+1+1) = 20 marks)

a) Copy and complete the following table:

Variable	Probability mass function or probability density function	Range	Generating function
$T \sim Bern(p)$	$P(T=k)=\begin{cases} p & k=1 \\ (1-p) & k=0 \\ 0 & \text{otherwise} \end{cases}$		
$X \sim Bin(n,p)$			$g_X(z)=[1-p+pz]^n$
$C \sim Geo(p)$			$g_C(z)=\dfrac{zp}{1-(1-p)z}$
$N \sim Poi(\lambda)$	$P(N=k)=\begin{cases} \dfrac{e^{-\lambda}\lambda^k}{k!} & k\in\{0,1,2,\ldots\} \\ 0 & \text{otherwise} \end{cases}$		
$W \sim \exp(\lambda)$	$f(w)=\begin{cases} \lambda e^{-\lambda w} & w\in[0,\infty) \\ 0 & \text{otherwise} \end{cases}$		$g_W(z)=\dfrac{\lambda}{\lambda-\ln(z)}$

b) Give an example of a possible random experiment and resulting variable such that the variable would have the following distribution:

i) $R \sim Bern\left(\dfrac{1}{13}\right)$;   ii) $S \sim Geo\left(\dfrac{1}{4}\right)$.

For example, rolling a regular fair six-sided die 20 independent times and counting how many times it lands on an even number would give a variable which $\sim Bin\left(20,\dfrac{1}{2}\right)$.

c) Let A and B be events in a sample space Ω such that

$P(A)=0.5$, $P(A\cap B)=0.1$ and $P(A\cup B)=0.6$.

Calculate:

i) $P(A^c)$;   ii) $P(B)$;   iii) $P(A|B)$;

iv) $P(B|A)$;   v) $P(B|A^c\cap B^c)$;   vi) $P((A\cup B)^c)$.

Let the event C in Ω be independent of A and such that B and C are mutually exclusive.

vii) Calculate $P(A\cap B\cap C)$.

Over...

Question 3 ((1+1+1+1+1+1+1+1+1+1+1) + (2+1+3+3) = 20 marks)

a) A hospital's administration office processes patient files 24 hours a day. On average, the office processes 10 files from male patients per hour and 10 files from female patients per hour. 30% of the files relate to child patients and 70% relate to adult patients. The office checks all files for errors while processing. 90% of files are classified as error-free and 10% are classed as containing errors.

Model this scenario by a Poisson process whereby patient files are processed independently of each other. Assume that the gender of each file's patient, the age of each file's patient and whether or not the file contains any errors are all independent.

Calculate the probabilities of the following events:

i) During a one hour period, exactly 15 files are processed;

ii) During a five minute period, no files containing errors are processed;

iii) During a ten minute period, no child files are processed, given no adult files are processed during that ten minutes;

iv) During the first two hours of the day, no files are processed from male adults and no files are processed from female children;

v) The difference between the time that the ninth file is processed and the time that the tenth file is processed is greater than ten minutes;

vi) The first file processed after 1p.m. is for a female patient and contains no errors;

vii) The last three files processed in a given day are all for child patients;

viii) Between 2:00pm and 3:00pm, exactly four files for female children are processed, given exactly three files for female children are processed between 2:00pm and 2:45pm;

ix) At least twenty files are processed in the first hour of the day, given that exactly twenty are processed in the first half hour of the day.

Find the distributions of the following variables:

x) Out of the next ten files processed, the number of files which are adult files containing errors;

xi) The time between the processing of the 60th file and the processing of the 61st file after a given timepoint.

Over...

Question 3 (Continued)

b) A game operates through a system of independent bets. At each turn, a player pays \$1. With probability p ($0 < p < 1$) he/she wins \$2 (i.e. his/her original stake plus \$1 of profit). With probability $q = 1 - p$ he/she loses his/her original stake and wins nothing.

He/she plays the game repeatedly until he/she reaches \$0 ("loses") or reaches \$$D$ ("wins"), whichever happens first.

Let W_k be the probability that he/she eventually wins given that, at a given time, he/she has \$$k$. (Assume k and D are integers and that $p \neq q$.)

i) Clearly explain why W_k satisfies $W_k = pW_{k+1} + qW_{k-1}$.

ii) Write down the boundary conditions, W_0 and W_D. Explain your answer.

iii) Solve the difference equation in part (i) to show that $W_k = \dfrac{1-\left(\dfrac{q}{p}\right)^k}{1-\left(\dfrac{q}{p}\right)^D}$.

He/she now doubles the amount staked on each bet. That is, with probability p, he/she gains \$2 profit and with probability $q = 1 - p$ suffers a \$2 loss.

Let $\tilde{W}_k$ be the probability that he/she eventually now wins given that, at a given time, he/she has \$$k$.

(You may assume that the player's initial wealth and D are the same as in parts i)-iii) and that both are divisible by 2.)

iv) Calculate $\tilde{W}_k$.

Over...

Question 4 ((2+2+2+2+2) + (2+2+4+2) = 20 marks)

a) Let X be a discrete random variable with probability mass function

$$P(X=k)=\begin{cases} 0.1 & k\in\{0,1,2,3\} \\ 0.2 & k\in\{5,6,7\} \\ 0 & \text{otherwise} \end{cases}$$

and let Y be a continuous random variable with probability mass function

$$f(y)=\begin{cases} \dfrac{y^4}{1555} & y\in[1,6] \\ 0 & \text{otherwise} \end{cases}$$

Verify that:

i) $\sum_k P(X=k)=1$;   ii) $\int_{-\infty}^{\infty} f(y)=1$.

Calculate:

iii) $E(X)$;   iv) $E(Y)$ v) $E\left(3Y-7X+\sin\left(\frac{1}{17}\right)\right)$.

b) i) For $W\sim\exp(\lambda)$ show, by differentiating the generating function, that $E(W)=\dfrac{1}{\lambda}$.

ii) For $C\sim Geo(p)$, by differentiating the generating function, calculate $E(C)$.

Let $W_1,W_2,W_3,\ldots$ be independent random variables such that each $W_i\sim\exp(\lambda)$ and let $C\sim Geo(p)$ be independent of each of these.

iii) Show that $V=\sum_{i=1}^{C}W_i$ is an exponential variable and find its rate parameter.

iv) Faults are reported to a 24 hour hotline such that the time (in days) between successive reports are independent variables, each $\sim\exp(20)$.
Each fault is assigned to one of 5 engineering teams, each with probability 0.2, independently of the arrival and assignment of all other faults.

Write down the expected time between successive faults assigned to one team.

Explain your answer in the context of your result for part iii)

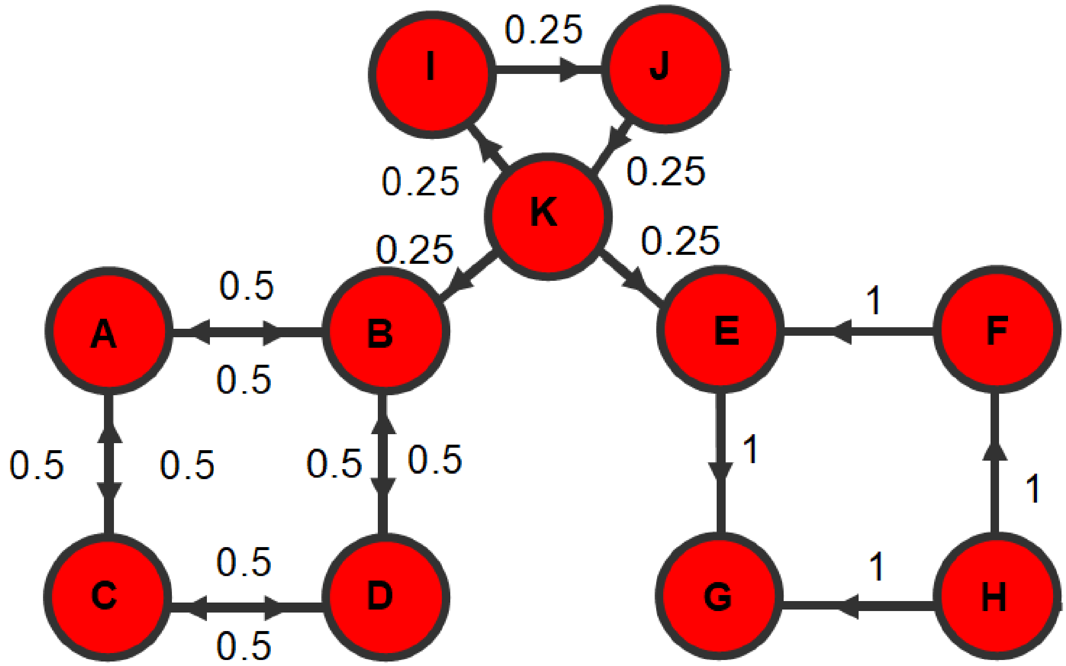
Hints: The generating functions of $W\sim\exp(\lambda)$ and $C\sim Geo(p)$ are given in Question 1.

You may use without proof the result that, if $V=\sum_{i=1}^{C}W_i$, $g_V(z)=g_C(g_{W_i}(z))$.

Over...

Question 5 ((4+2+2+2+2) + (3+2+3) = 20 marks)

a) Let $X_0, X_1, X_2,\ldots$ be a Markov Chain which is represented by the state diagram



i) Find the transition matrix for this chain, with the columns/rows corresponding to the states in alphabetical order, from A to K.

An absorbing state is one such that, if the system ever enters that state, the probability that it is ever in a different state is zero.

A persistent state is one such that, if the system is ever in that state and moves to another state, the probability that it never returns is zero.

A transient state is one such that, if the system is ever in that state and moves to another state, the probability that it eventually returns is less than one.

A state i is periodic with period $d > 1$ if $P(X_{n+k} = i \mid X_n = i) = 0$ for all n unless k is divisible by d.

Which of the states for this chain are:

ii) absorbing; iii) persistent; iv) transient?

v) Which states are periodic with period $d > 1$? For each of these states, find the period.

Over…

Question 5 (Continued)

b) Let $Y_0, Y_1, Y_2, \ldots$ be a Markov Chain with transition matrix $P = [p_{ij}] = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \end{pmatrix}$

where $p_{ij} = P(Y_{n+1} = j \mid Y_n = i)$

i) Draw the state diagram for this chain.

ii) Calculate the 2-step transition probability $P(Y_{n+2} = 2 \mid Y_n = 3)$.

iii) Calculate the equilibrium distribution for this chain.

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