37161 Probability and Random Variables

Lecture 10

Markov Chains

- Recall the definition that a sequence of random variables, $\{X_0, X_1, X_2, \ldots\}$ each with range S was a Markov Chain only if $P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots X_0 = i_0) = P(X_{n+1} = j \mid X_n = i)$ for all possible n and all possible $j, i, i_{n-1}, i_{n-2}, \ldots, i_0 \in S$.

- In other words, when predicting X_{n+1} given the values of $\{X_0, X_1, X_2, \ldots, X_n\}$, for a Markov Chain, we only need to know the value of X_n .

- Viewing the sequence as observations through time, this is sometimes stated as "given the present value, the future results are independent of the past."

Markov Chains: Transition Probabilities

- For a Markov Chain, $\{X_{0,} X_{1,} X_{2,} \ldots\}$ we define a **transition probability** to be the conditional probability that X_n takes the value *j*, given that X_{n-1} takes the value *i*.

- These probabilities are typically written with subscripts e.g. $p_{ij}(n) = P(X_n = j \mid X_{n-1} = i)$.

- If the transition probabilities do not depend upon *n*, then we say that the probabilities are **stationary**.

- In this subject, we will only look at homogeneous Markov Chains, which are ones where all transition probabilities are stationary.

- In such cases, there is no need to include dependence on *n*, i.e. we would simply write $p_{ij} = P(X_n = j \mid X_{n-1} = i)$.

Transition Probabilities: Properties

- The transition probabilities have to be valid probabilities, i.e. for any system $0 \leq p_{ij} \leq 1$ for all possible *i* and *j*.
- Similarly, given that $X_{n-1} = i$, the next variable in the sequence has to take some value from *S* with probability 1.
- This gives $\sum_{j \in S} P(X_n = j | X_{n-1} = i) = 1$ or, alternatively $\sum_{j \in S} p_{ij} = 1$.
- Usually, for finite Markov Chains (i.e. those whose variables can only take one of a finite number of values), the transition probabilities are written in a **transition matrix**.

Transition Matrices

- The transition matrix of a finite Markov Chain is defined as

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1(n-1)} & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2(n-1)} & p_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{(n-1)1} & p_{(n-1)2} & \cdots & p_{(n-1)(n-1)} & p_{(n-1)n} \\ p_{n1} & p_{n2} & \cdots & p_{n(n-1)} & p_{nn} \end{pmatrix}$$

- Note that since for all possible *i* and *j*, $\sum_{j \in S} p_{ij} = 1$, the sum across each row must equal 1.
- The sum down each column, however, does not need to equal 1 (although it might.)

Transition Matrices: Example

- Five children are playing Pass the Parcel.

- Let the children be numbered 1, 2, 3, 4 and 5 and are seated sequentially in a circle (and with 5 next to 1.)

- At any given time, exactly one of the children has hold of the parcel. Each child can only receive it from the child numbered one lower (except 1, who receives it from 5) and can only pass it to the child numbered one higher (apart from 5 who gives it to 1.)

Transition Matrices: Example

- Let the number of the child holding the parcel after *n* seconds be X_n . Assuming each child does not change his/her behaviour depending on how the other children have acted, then the sequence of the parcel's position each second forms a Markov Chain.

- Assume that Child 1 – Child 4 all behave the same but that Child 5 behaves differently.

Transition Matrices: Example

- For Child 1 – Child 4, if he/she is holding the parcel after *n* seconds, there is a 50% chance that he/she will still be holding it after *n* +1 seconds and a 50% chance that it be with the child numbered one higher.

- Child 5 holds the parcel a little longer each time. If he/she is holding it after *n* seconds, there is a 75% chance that he/she still has it after *n* +1 seconds and a 25% chance that it is now with child 1.

- The transition matrix is therefore $P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0 & 0 & 0 & 0.75 \end{pmatrix}$

Transition Matrices

- How can the transition matrix be used to calculate the probabilities of possible future states?
- For each *j*, we have that $P(X_{n+1} = j) = \sum_{i=1}^{5} P(X_n = i) \times p_{ij}$. In other words, the probability that child *j* has the parcel after *n* + 1 seconds is equal to the probability that child 1 has it after *n* seconds multiplied by the probability that it is passed from 1 to *j* during the next second plus the probability that child 2 has it after *n* seconds multiplied by the probability that it is passed from 2 to *j* during the next second etc.
- Let $\Pi_n = (\pi_n^1 \quad \pi_n^2 \quad \pi_n^3 \quad \pi_n^4 \quad \pi_n^5)$ be a row vector of the probabilities that the parcel is in the hands of each child at time *n*, where $\pi_n^1 = P(X_n = 1)$.
- For transition matrix *P*, we have $\Pi_n P = \Pi_{n+1}$.

Transition Matrices: Example

- Assume that that one child is selected at random to have the parcel after 0 seconds, with each child equally likely to be chosen. Here, we have $\Pi_0 = (0.2 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0.2)$.

- What is the probability distribution of the parcel's position after 1 second?

- $\Pi_1 = \Pi_0 P = (0.2 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0.2)\begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0 & 0 & 0 & 0.75 \end{pmatrix} = (0.15 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0.25)$

- In other words, it is now more likely to be in the hands of child 5 than any other, which is expected, given that he/she holds it longer.

Transition Matrices: *n*-Step Transitions

- Calculation of the distribution of states more than one step into the future is similar.
- We know that $\Pi_{n+1} = \Pi_n P$, so $\Pi_{n+2} = \Pi_{n+1} P$ and hence $\Pi_{n+2} = (\Pi_n P)P = \Pi_n P^2$.
- For the Pass the Parcel example, we have

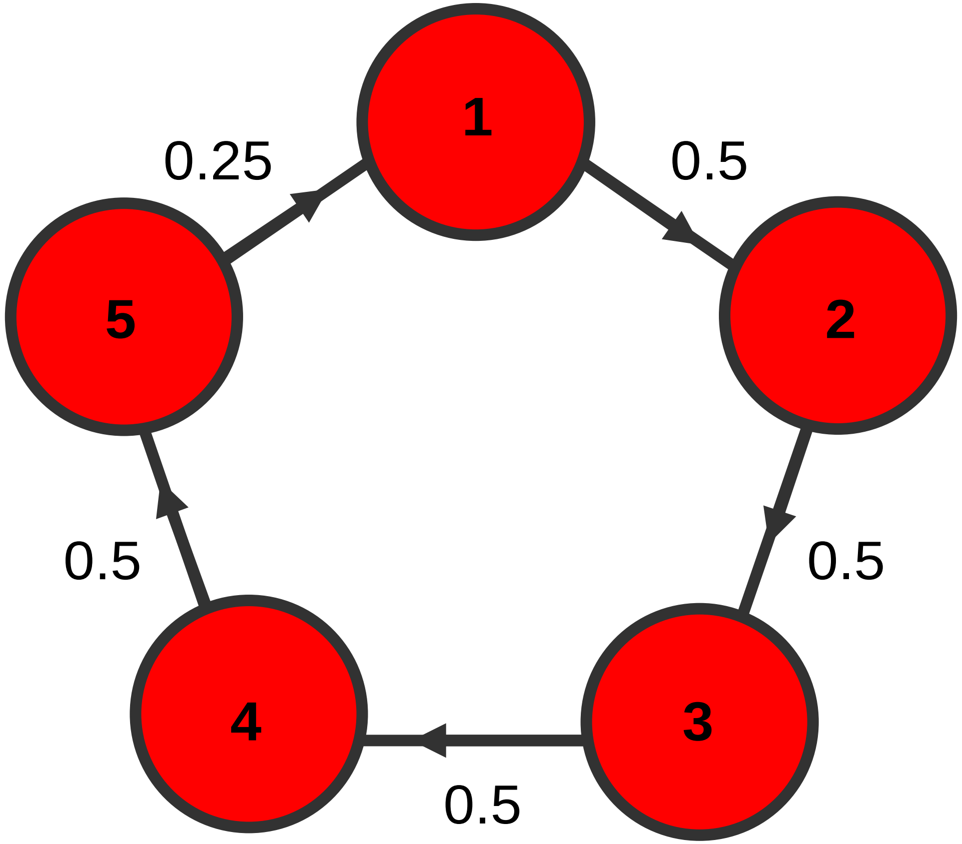
$$\Pi_2 = \Pi_0 P^2 = \Pi_1 P = (0.15 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0.25)\begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0 & 0 & 0 & 0.75 \end{pmatrix}$$

$$= (0.1375 \quad 0.175 \quad 0.2 \quad 0.2 \quad 0.2875)$$

- We call the matrix for calculating *n* steps into the future, the ***n*-step transition matrix**.

State Diagrams

- When examining long-term behaviour of a Markov Chain, it can sometimes help to visualise the transitions between states on a **state diagram**.
- Each possible state is represented by a node on a graph and nodes are connected by directed, weighted arrows indicating the probabilities of moving from each state to each other (by convention, arrows corresponding to transitions with probability 0 are left out.)
- For example, the state diagram for the Pass the Parcel example is:



Steady-State Distributions

- Consider now a Markov Chain which has been running for a long time.
- Many such systems settle down to give a **steady-state** or **equilibrium distribution** such that, as $n \to \infty$, the probability of the system being in each state at any future time tends to some constant.
- That is $\Pi_{n+1} - \Pi_n \to 0$.
- Let $\lim_{n\to\infty} \Pi_n = \lim_{n\to\infty} \Pi_{n+1} = \Pi_{eq}$. We find Π_{eq} by letting $n \to \infty$ in $\Pi_n P = \Pi_{n+1}$ i.e. we solve $\Pi_{eq} P = \Pi_{eq}$
- Π_{eq} is therefore found by finding the vector such that $\Pi_{eq} P = \Pi_{eq}$, subject to the fact that the sum of all probabilities in Π_{eq} must equal 1.

Steady-State Distributions: Example

- For the Pass the Parcel example, we have $P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0 & 0 & 0 & 0.75 \end{pmatrix}$
- To find Π_{eq} such that $\Pi_{eq} P = \Pi_{eq}$, let $\Pi_{eq} = \begin{pmatrix} A & B & C & D & E \end{pmatrix}$ and solve subject to $A + B + C + D + E = 1$.

Steady-State Distributions: Example

- This gives

$$\begin{pmatrix} A & B & C & D & E \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0 & 0 & 0 & 0.75 \end{pmatrix} = \begin{pmatrix} \frac{2A+E}{4} & \frac{2A+2B}{4} & \frac{2B+2C}{4} & \frac{2C+2D}{4} & \frac{2D+3E}{4} \end{pmatrix}$$

subject to $A+B+C+D+E=1$.

- This can be solved by $2A=2B=2C=2D=E$.

- Ensuring that $A+B+C+D+E=1$ we obtain $\Pi_{eq} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$.

Steady-State Distributions: Eigenvectors

- Note that finding equilibrium distributions is equivalent to finding eigenvectors.
- Transposing $\Pi_{eq} = \Pi_{eq} P$ gives $\Pi_{eq}{}^t = P^t \Pi_{eq}{}^t$.
- Recall that ***v*** was an eigenvector of matrix ***A*** corresponding to the eigenvalue λ if and only if ***Av*** $= \lambda$ ***v***.
- The steady-state distribution Π_{eq} can is therefore the transpose of the eigenvector of P^t corresponding to the eigenvalue 1.
- Note in some cases, there can be more than one linearly independent eigenvector corresponding to 1 (if the characteristic equation det(***A*** – λ ***I***) = 0 has a repeated root of $\lambda = 1$.)

Steady-State Distributions: Eigenvectors

- For a (correctly defined) transition matrix P, $\lambda = 1$ is always an eigenvalue of P^t.
- This is because the rows of P each sum to 1, $\sum_{j \in S} p_{ij} = 1$.
- Once transposed, this guarantees that each of the columns of P^t sums to 1.
- Subtracting 1 from the elements down the leading diagonal ensures that det(P^t – $\boldsymbol{I}$) = 0 since each column in P^t – $\boldsymbol{I}$ now sums to zero, ensuring that the determinant is zero.

Steady-State Distributions: Eigenvectors

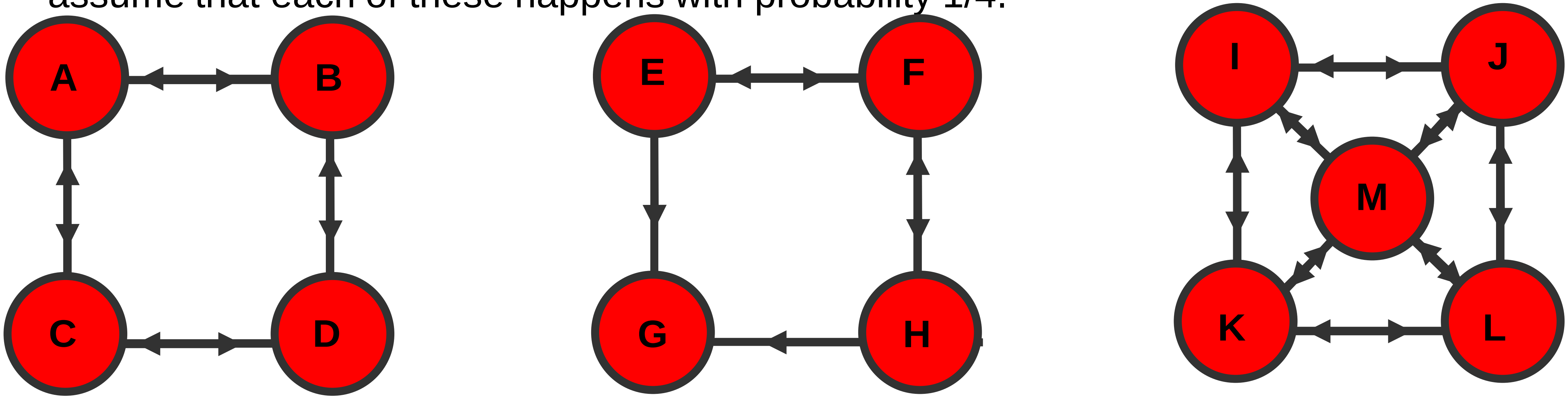
- Especially for high-dimensional matrices, finding eigenvectors (and hence the steady-state distributions) is not a complex task, but it can be extremely time-consuming.

- If you have access to a mathematical package to solve for eigenvectors, this can be enormously beneficial, but you will need to remember to scale the eigenvector such that the elements all sum to 1.

- In many cases, though, a simple look at the state diagram will give some hints as to what the equilibrium distribution might be.

Steady-State Distributions: Examples

- For each of the following examples, assume that all possible transitions from a state are equally likely. For example, if there are 4 possible states to move to from your current position, assume that each of these happens with probability 1/4.

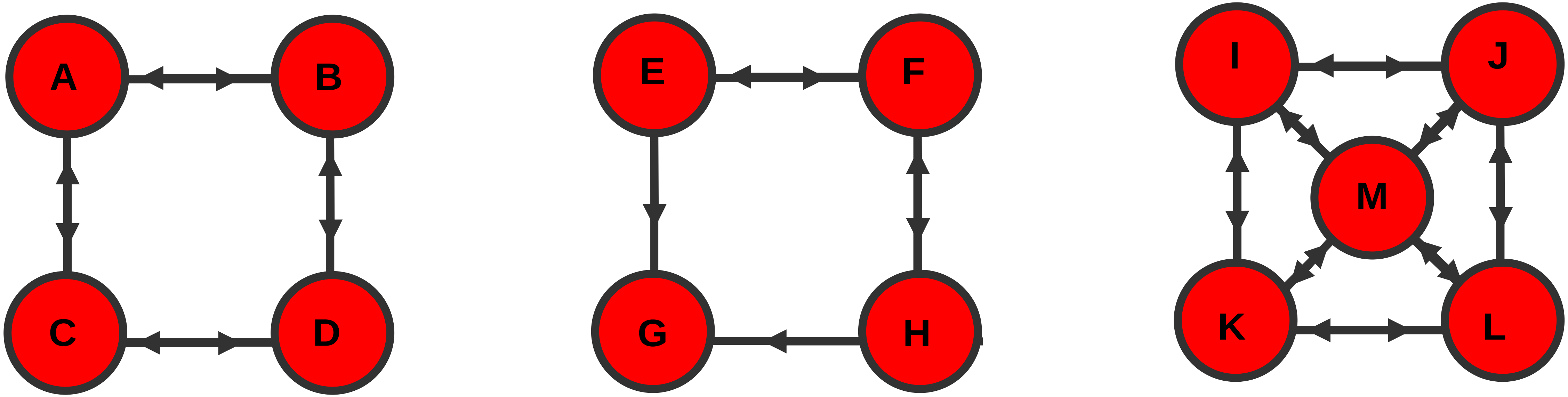


- If each system is left for a very long time (moves $\rightarrow \infty$) what are the equilibrium probabilities of the systems being in each possible state?

Steady-State Distributions: Examples

- The first two systems are easy to analyse. ABCD is completely symmetric, hence

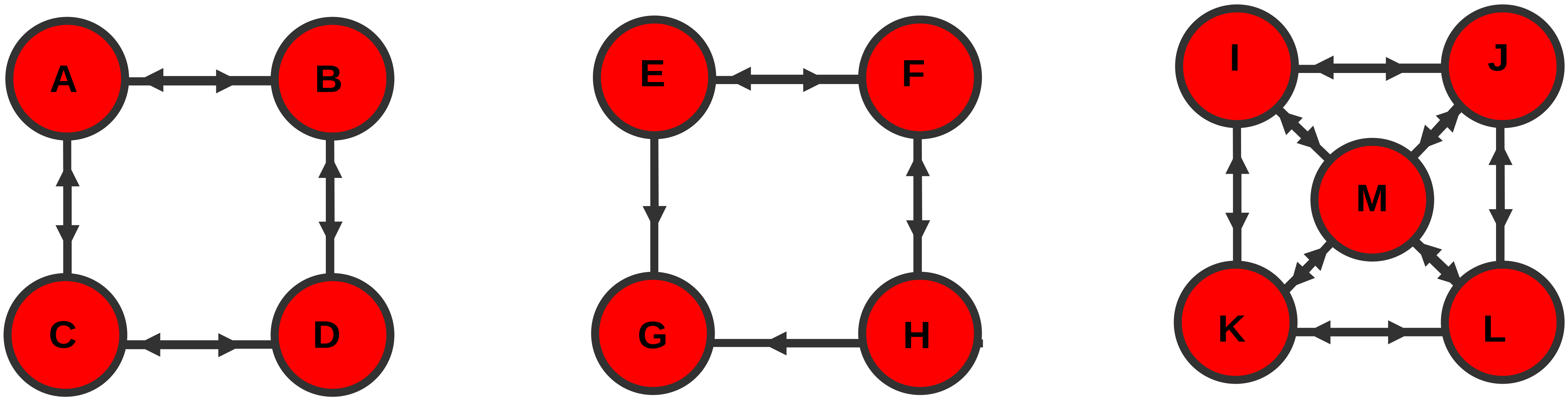
$$\Pi_{eq} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}$$



- Similarly, we can see that once the EFGH system lands in state G, it cannot leave hence eventually $\Pi_{eq} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$

Steady-State Distributions: Examples

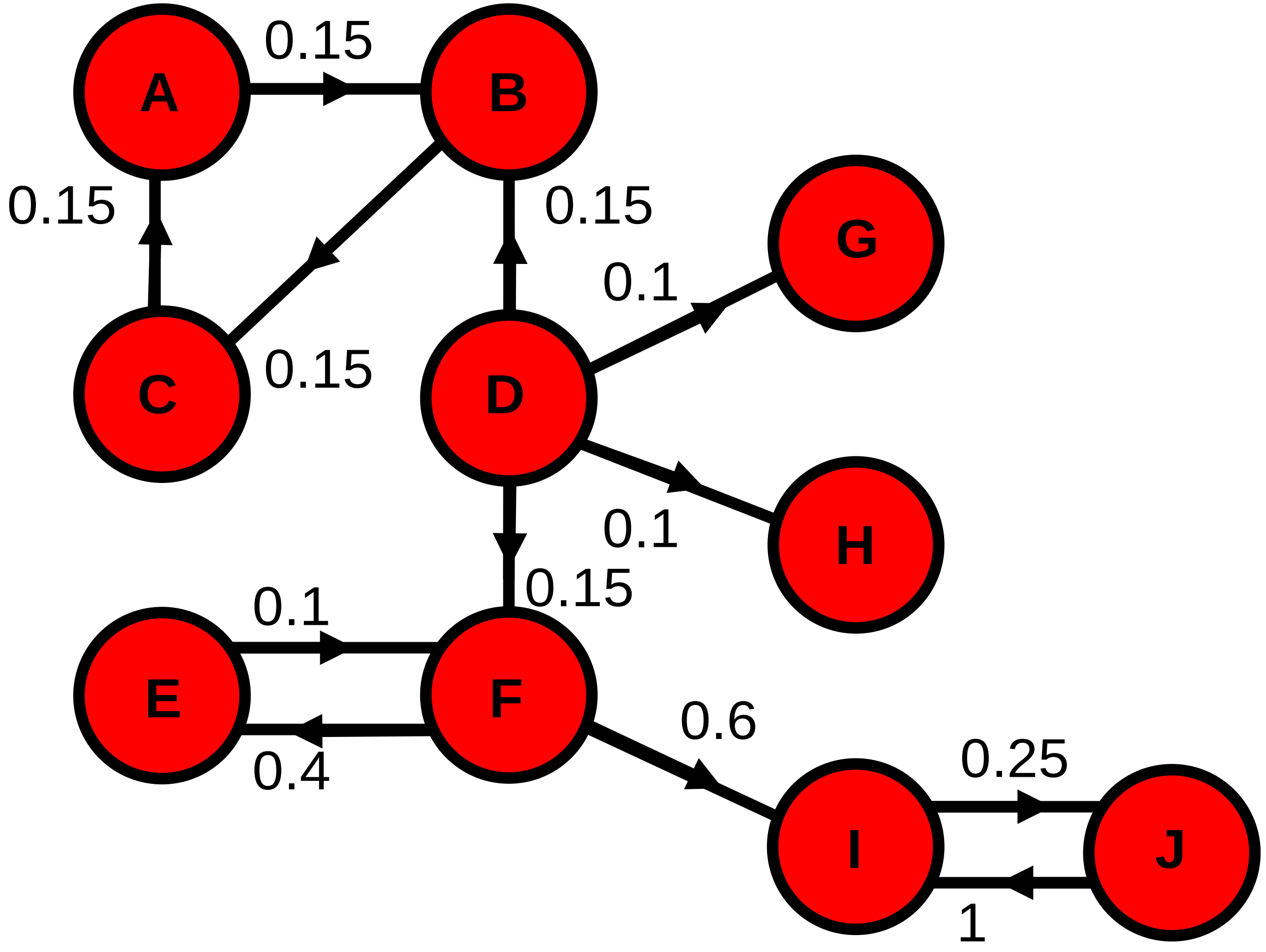
- The IJKLM system is much less intuitive, although the fact that I, J, K and L are all equally likely is perhaps obvious, as is the fact that M is a little more likely.



- In fact, $\Pi_{eq} = \begin{pmatrix} \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} & \frac{1}{4} \end{pmatrix}.$

Steady-State Distributions: Examples

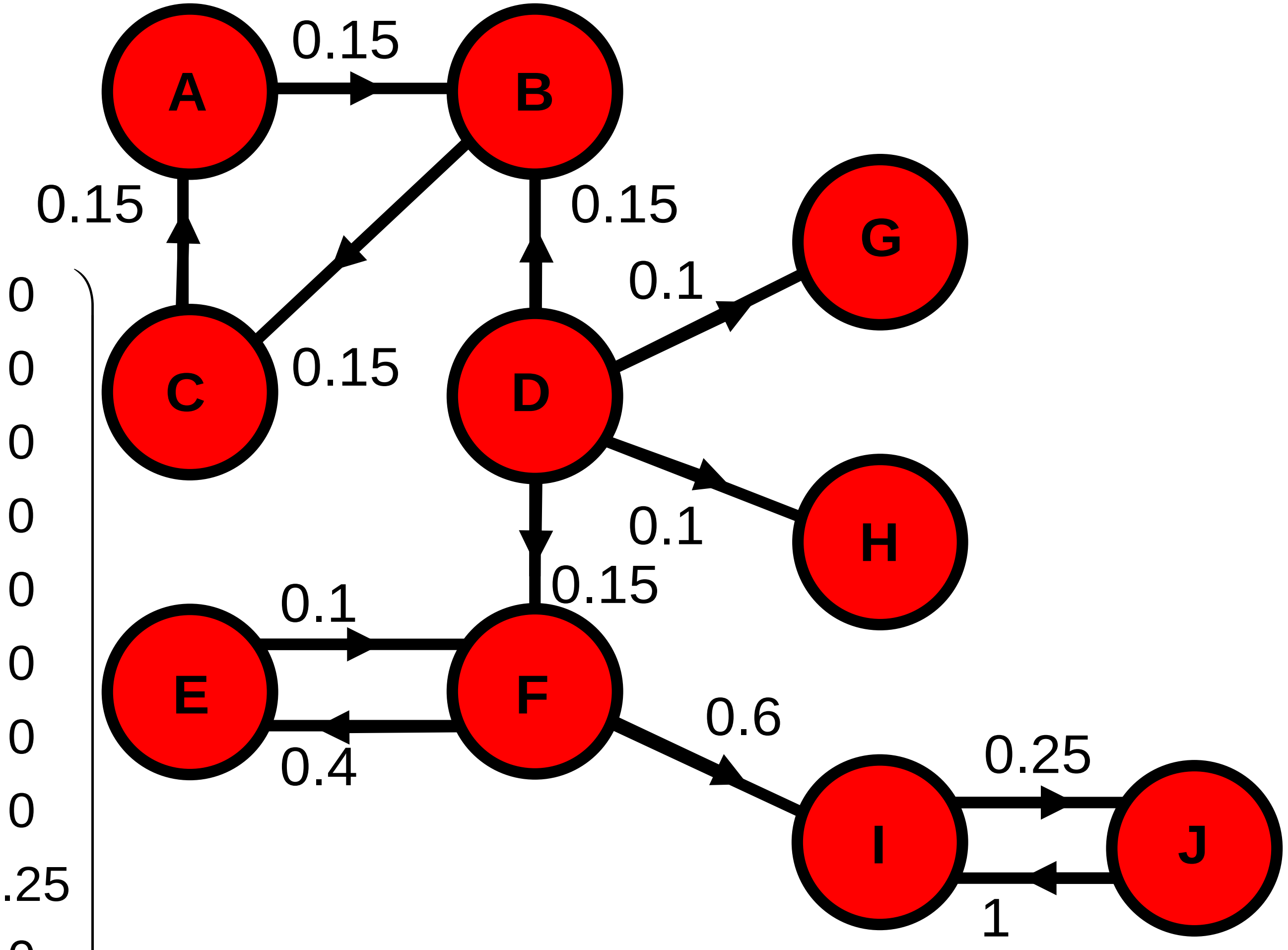
- Consider the Markov chain with this state diagram.
- How many (linearly independent) equilibrium distributions does it have?
- How can we find these?



Steady-State Distributions: Examples

- We can write down the transition matrix.

$$
\begin{pmatrix}
0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$



Steady-State Distributions: Examples

- Given the transition matrix, we can find solutions of the form

$$\begin{pmatrix}\pi^A & \pi^B & \pi^C & \pi^D & \pi^E & \pi^F & \pi^G & \pi^H & \pi^I & \pi^J\end{pmatrix}\begin{pmatrix}0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}\pi^A & \pi^B & \pi^C & \pi^D & \pi^E & \pi^F & \pi^G & \pi^H & \pi^I & \pi^J\end{pmatrix}$$

subject to $\sum \pi^k = 1.$

- Finding 10 dimensional eigenvectors is, of course, not a simple task.

- We can, though, make the problem easier by first analysing the state diagram and finding equilibrium distributions qualitatively.

Steady-State Distributions: Examples

- Analysing the state diagram, we can see that there are four (linearly independent) equilibrium distributions

- Cycling around states A, B and C

or

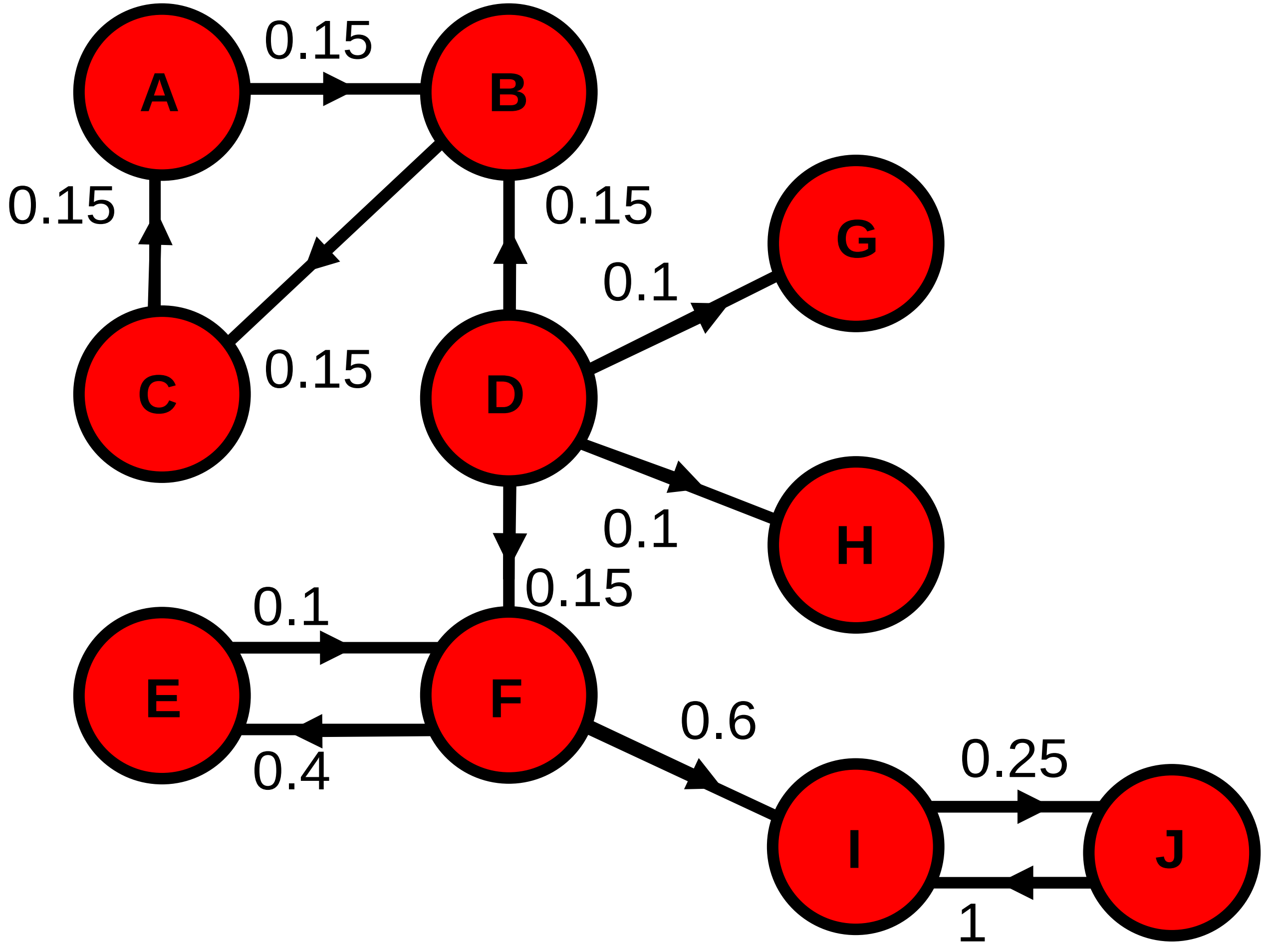
- Stuck in state G

or

- Stuck in stage H

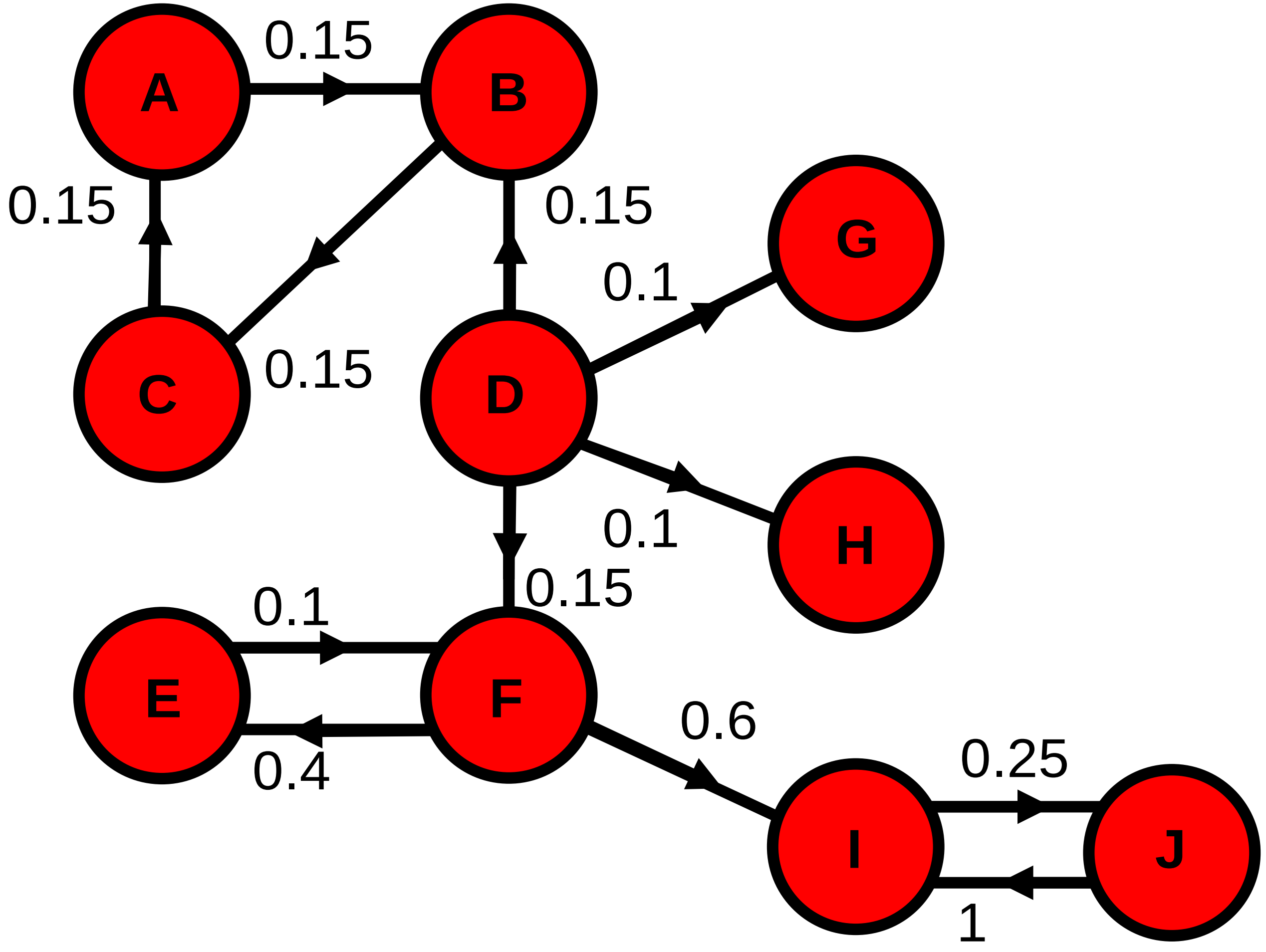
or

- Cycling around states I and J.



Steady-State Distributions: Examples

- We can therefore look for quantitative solutions which satisfy:



- Cycling around states A, B and C

or

- Stuck in state G

or

- Stuck in stage H

or

- Cycling around states I and J.

Steady-State Distributions: Examples

$$\begin{pmatrix}\pi^A & \pi^B & \pi^C & \pi^D & \pi^E & \pi^F & \pi^G & \pi^H & \pi^I & \pi^J\end{pmatrix}\begin{pmatrix}0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix}\pi^A & \pi^B & \pi^C & \pi^D & \pi^E & \pi^F & \pi^G & \pi^H & \pi^I & \pi^J\end{pmatrix}$$

subject to $\sum \pi^k = 1.$

- Solving to find the solution corresponding to cycling around states A, B and C, we can set all probabilities to zero apart from π^A, π^B and π^C.

Steady-State Distributions: Examples

$$\begin{pmatrix}\pi^A & \pi^B & \pi^C & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix}\pi^A & \pi^B & \pi^C & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

subject to $\sum \pi^k = 1.$

- We can easily see that

$$\pi^A = 0.85\pi^A + 0.15\pi^C$$
$$\pi^B = 0.15\pi^A + 0.85\pi^B + 0.15\pi^D$$
$$\pi^C = 0.15\pi^B + 0.85\pi^C$$
$$\pi^D = 0.5\pi^D$$

solve to give $\pi^A = \pi^B = \pi^C = \frac{1}{3}$ and $\pi^D = 0.$

Steady-State Distributions: Examples

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \pi^G & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \pi^G & 0 & 0 & 0 \end{pmatrix}$$

subject to $\sum \pi^k = 1.$

- Solving to find the solution corresponding to being stuck in state G, we can set all probabilities to zero apart from π^G.

- Trivially, this gives $\pi^G = 1.$

Steady-State Distributions: Examples

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pi^H & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pi^H & 0 & 0 \end{pmatrix}$$

subject to $\sum \pi^k = 1$.

- Solving to find the solution corresponding to being stuck in state H, we can set all probabilities to zero apart from π^H.
- Trivially, this gives $\pi^H = 1$.

Steady-State Distributions: Examples

$$\begin{pmatrix}\pi^A & \pi^B & \pi^C & \pi^D & \pi^E & \pi^F & \pi^G & \pi^H & \pi^I & \pi^J\end{pmatrix}\begin{pmatrix}0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

$$= \begin{pmatrix}\pi^A & \pi^B & \pi^C & \pi^D & \pi^E & \pi^F & \pi^G & \pi^H & \pi^I & \pi^J\end{pmatrix}$$

subject to $\sum \pi^k = 1.$

- Solving to find the solution corresponding to cycling around states I and J, we can set all probabilities to zero apart from π^I and π^J.

Steady-State Distributions: Examples

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pi^I & \pi^J \end{pmatrix} \begin{pmatrix} 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.15 & 0 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0.5 & 0 & 0.15 & 0.1 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0.25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pi^I & \pi^J \end{pmatrix}$$

subject to $\sum \pi^k = 1$.

- We can easily see that

$$\pi^I = 0.6\pi^F + 0.75\pi^I + \pi^J$$

$$\pi^J = 0.25\pi^I$$

solve to give $\pi^I = \frac{4}{5}$ and $\pi^J = \frac{1}{5}$.

Steady-State Distributions: Examples

- The equilibrium distributions are therefore given by:

- Cycling around states A, B and C

or $\Pi_{eq} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

- Stuck in state G

or $\Pi_{eq} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

- Stuck in stage H

or $\Pi_{eq} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

- Cycling around states I and J.

$\Pi_{eq} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} \end{pmatrix}$