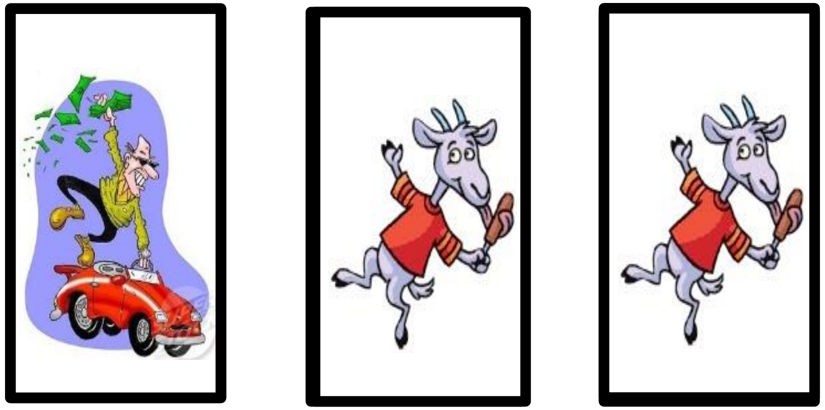
37161 Probability and Random Variables

Lecture 3

An “Average” Outcome

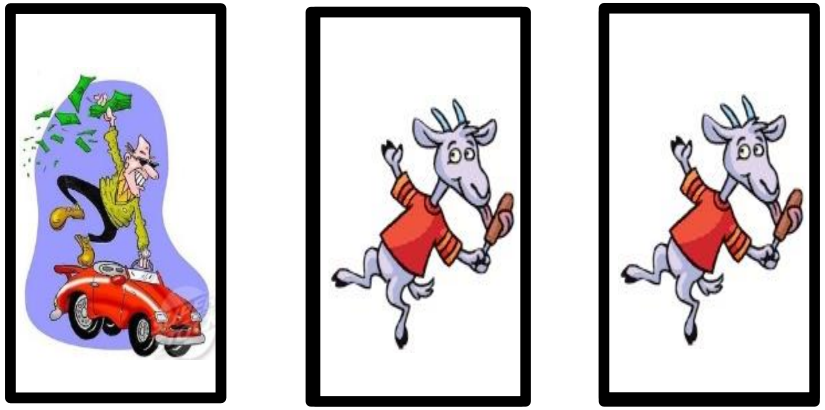


- Think back to the Monty Hall problem, as outlined in Lecture 2.

- In the initial setup, two of the three boxes contain a goat and the other box contains a sportscar.

- “On average” (whatever that means…) what is in the box which the contestant first selects?

An “Average” Outcome



- “On average” (whatever that means…) what is in the box which the contestant first selects?

- This is very difficult to define as, for a large number of independent trials, 1/3 of the time, the box will contain a sportscar and 2/3 of the time, it will contain a goat.

- We can’t meaningfully calculate “values” such as $\frac{1}{3} \times \text{sportscar} + \frac{2}{3} \times \text{goat}.$

- If the random experiment contains real numerical values, then the calculation of what we can expect on average is more meaningful.

Expected Outcomes

- For example, if we are given the choice of two sealed envelopes, one containing \$5 and the other containing \$10, it is easily seen that, if played a large number of independent times, we should get an average of \$7.50 per game.

- This is because half of the games will end with a \$5 and half with a \$10 win, so the average is $\frac{1}{2}\times\$5 + \frac{1}{2}\times\$10 = \$7.50.$

- Similarly, we could give an expected value for the outcome of the Monty Hall problem if we allocated some numerical value (financial worth?) to both the sportscar and the goats.

Random Variables

- A random variable is a function which maps all possible outputs Ω of a random experiment to some subset of $\mathbb{R}$.

- In other words, it takes the outcome of a given experiment (which could be numerical, or could be a category e.g. “sportscar” or “goat”) and assigns a real number to it.

- For experiments whose sample space is numerical values, a random variable can simply defined as the number of the event in the sample space.

Probability Mass Functions

- For a discrete random variable, the probability mass function gives, for all real numbers, the probability that the random variable gives that numerical value.
- For example, if we are rolling one regular fair six-sided die and defining the random variable X to be the number shown, then we have that all values 1,2,…,6 occur with probability 1/6 and that no other values can occur.
- This is $P(X=k)=\begin{cases} 1/6 & k=1 \\ 1/6 & k=2 \\ 1/6 & k=3 \\ 1/6 & k=4 \\ 1/6 & k=5 \\ 1/6 & k=6 \\ 0 & \text{otherwise} \end{cases}$

Probability Mass Functions

- Consider now rolling one regular fair six-sided die and defining the random variable Y to be the number of letters in the name of the numbers shown (ONE, TWO, THREE, FOUR, FIVE or SIX).

- Three of the (equally probable outcomes) have 3 letters, so there is a 3/6 or 1/2 chance that Y will take the value 3.

- Similarly, there is a 1/3 chance of getting a number with four letters and a 1/6 chance of getting a five letter number.

- Overall, we get $P(Y = k) = \begin{cases} 1/2 & k = 3 \\ 1/3 & k = 4 \\ 1/6 & k = 5 \\ 0 & \text{otherwise} \end{cases}$

Expectation of Discrete Random Variables

- We can now define the expectation of a discrete random variable.

- Consider a simple coin flip game whereby a player wins $1 if the coin lands Heads and loses $1 if the coin lands tails. If this game is played a large number of times, he/she can expect to gain $1 in half of his/her games and can expect to lose $1 in the other half.

- On average, he/she will not expect to be gaining or losing money as the game is neither biased for nor against the player.

Expectation of Discrete Random Variables

- We define the expectation of a random variable X with probability mass function $P(X = k)$ as $\sum (k \times P(X = k))$ where the summation includes all possible values of k.

- Common notation will often write $E(X) = \mu$.

Expectation of Discrete Random Variables

- What is the expected number shown by one roll of a regular fair six-sided die?

- Let the number shown be X. We have already seen that this has probability mass function

$$P(X=k)=\begin{cases} 1/6 & k=1 \\ 1/6 & k=2 \\ 1/6 & k=3 \\ 1/6 & k=4 \\ 1/6 & k=5 \\ 1/6 & k=6 \\ 0 & \text{otherwise} \end{cases}$$

- The expectation is therefore

$$E(X)=\left(\frac{1}{6}\times 1\right)+\left(\frac{1}{6}\times 2\right)+\left(\frac{1}{6}\times 3\right)+\left(\frac{1}{6}\times 4\right)+\left(\frac{1}{6}\times 5\right)+\left(\frac{1}{6}\times 6\right)=\frac{21}{6}=3.5$$

- The expected number shown by one die is therefore 3.5.

Option Pricing

- A **call option** is a financial contract which gives someone the right (but not responsibility) to purchase a particular commodity or stock on a given date (**expiry date**) for a fixed price (**strike price**).

- If the value of that stock is no higher than the strike price, the option holder can simply purchase it at its current price. If the value is higher than the strike price, the holder can purchase it cheaper at the strike price.

- Technically this is known as the **European call option**. An **American call option** allows the holder to purchase at the strike price at any time at or before the expiry date for the strike price.
- A similar option with an agreed sale price is known as a **put option**.

Option Pricing

- Assume that you are offered a call option to purchase stock in a particular company for \$7 per share at the end of the year.

- What would be a fair price to pay for this option?

- It depends, of course, on how you believe the share price will change between now and the end of the year.

- Taking the (unrealistically naïve) model that the share price at the end of the year will be either \$1, \$2, …, or \$10, each with probability 10%, at what price does the option give you expected profits?

Option Pricing

- If the year end share price is no more than $7, then the option is worthless.

- If the year end share price is $8, the contract is worth $1 profit per share.

- If the year end share price is $9, the contract is worth $2 profit per share. Similarly, if it is $10, then the contract is worth $3 profit per share.
- The expected value of the option per share is therefore
$$\left(\frac{7}{10}\times\$0\right)+\left(\frac{1}{10}\times\$1\right)+\left(\frac{1}{10}\times\$2\right)+\left(\frac{1}{10}\times\$3\right)=\$0.60$$
- This is the basis of many financial trades (albeit with much more complex and realistic models.)

How Many Draws?

- Consider drawing one card at random from a standard deck of 52, with each card equally likely to be selected.

- How many cards would you expect to have to draw until you first selected a King, assuming that after each draw you placed your card back into the deck and reshuffled?

How Many Draws?

- The probability that you draw your first King on your first draw is $\frac{4}{52}$ or $\frac{1}{13}$.
- The probability that you draw your first King on your second draw is $\frac{12}{13} \times \frac{1}{13}$, since this happens only if you don't draw a King first time, but you do second time.
- Likewise, probability that you draw your first King on your third draw is $\frac{12}{13} \times \frac{12}{13} \times \frac{1}{13}$.

How Many Draws?

- In general, the probability that a total of $k \geq 1$ cards are drawn before the first King is selected is $\frac{1}{13}\left(\frac{12}{13}\right)^{k-1}$, since we need the first $k-1$ cards to be non-Kings, and the kth to be a King.
- We therefore have $P(X=k)=\begin{cases}\frac{1}{13}\left(\frac{12}{13}\right)^{k-1} & k=1,2,3,\ldots \\ 0 & \text{otherwise}\end{cases}$
- The expectation of this is therefore

$$E(X)=\left(1\times\frac{1}{13}\right)+\left(2\times\frac{1}{13}\left(\frac{12}{13}\right)\right)+\left(3\times\frac{1}{13}\left(\frac{12}{13}\right)^2\right)+\left(4\times\frac{1}{13}\left(\frac{12}{13}\right)^3\right)+\left(5\times\frac{1}{13}\left(\frac{12}{13}\right)^4\right)+\ldots$$

How Many Draws?

- We can evaluate $E(X)$ through geometric series. We know that, if $-1<r<1$, then

$$A + Ar + Ar^2 + Ar^3 + Ar^4 + \ldots = \frac{A}{1-r}$$

- To do this, we rewrite

$$E(X) = \left(1\times\frac{1}{13}\right) + \left(2\times\frac{1}{13}\left(\frac{12}{13}\right)\right) + \left(3\times\frac{1}{13}\left(\frac{12}{13}\right)^2\right) + \left(4\times\frac{1}{13}\left(\frac{12}{13}\right)^3\right) + \left(5\times\frac{1}{13}\left(\frac{12}{13}\right)^4\right) + \ldots$$

as a geometric series of individual geometric series.

How Many Draws?

$$E(X) = \left(1 \times \frac{1}{13}\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^2\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^3\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^4\right) + \ldots \qquad 1$$

$$+ \left(\frac{1}{13}\left(\frac{12}{13}\right)\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^2\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^3\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^4\right) + \ldots \qquad + \left(\frac{12}{13}\right)$$

$$+ \left(\frac{1}{13}\left(\frac{12}{13}\right)^2\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^3\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^4\right) + \ldots \qquad + \left(\frac{12}{13}\right)^2$$

$$+ \left(\frac{1}{13}\left(\frac{12}{13}\right)^3\right) + \left(\frac{1}{13}\left(\frac{12}{13}\right)^4\right) + \ldots \qquad + \left(\frac{12}{13}\right)^3$$

$$+$$

- Summing across each row, we get row totals which form a geometric series. $\ldots$

How Many Draws?

- $$E(X)=\left(1\times\frac{1}{13}\right)+\left(\frac{1}{13}\left(\frac{12}{13}\right)\right)+\left(\frac{1}{13}\left(\frac{12}{13}\right)^2\right)+\left(\frac{1}{13}\left(\frac{12}{13}\right)^3\right)+\left(\frac{1}{13}\left(\frac{12}{13}\right)^4\right)+\ldots$$
$$+\left(\frac{1}{13}\left(\frac{12}{13}\right)\right)+\left(\frac{1}{13}\left(\frac{12}{13}\right)^2\right)+\left(\frac{1}{13}\left(\frac{12}{13}\right)^3\right)+\left(\frac{1}{13}\left(\frac{12}{13}\right)^4\right)+\ldots$$
- e.g. the top line has first term $A=\frac{1}{13}$ and common ratio $r=\frac{12}{13}$. Its sum is $\frac{\frac{1}{13}}{1-\frac{12}{13}}=1$
- The next line has first term $A=\frac{1}{13}\left(\frac{12}{13}\right)$ common ratio $r=\frac{12}{13}$. Its sum is $\frac{\left(\frac{12}{13}\right)\frac{1}{13}}{1-\frac{12}{13}}=\frac{12}{13}$

How Many Draws?

- The expected number of draws is therefore $E(X) = 1 + \left(\frac{12}{13}\right) + \left(\frac{12}{13}\right)^2 + \left(\frac{12}{13}\right)^3 + \left(\frac{12}{13}\right)^4 + \ldots$

- This itself is a geometric series, first term $A = 1$, common ratio $r = \frac{12}{13}$.

- The expected number of draws until the first King is therefore $E(X) = \frac{1}{1 - \frac{12}{13}} = 13$

- This is perhaps intuitive, since if Kings occur, on average, on $\frac{1}{13}$ of the draws, then we can expect to wait around 13 draws until first selecting a King.

St Petersburg Paradox

- Consider a game that consists of paying a fee of $\$M$ to flip one fair coin. If the coin lands Heads, the player gets nothing back (i.e. loses his/her money). If the coin lands Tails, he/she gets $\$2M$ back (i.e. wins an additional $\$M$.)

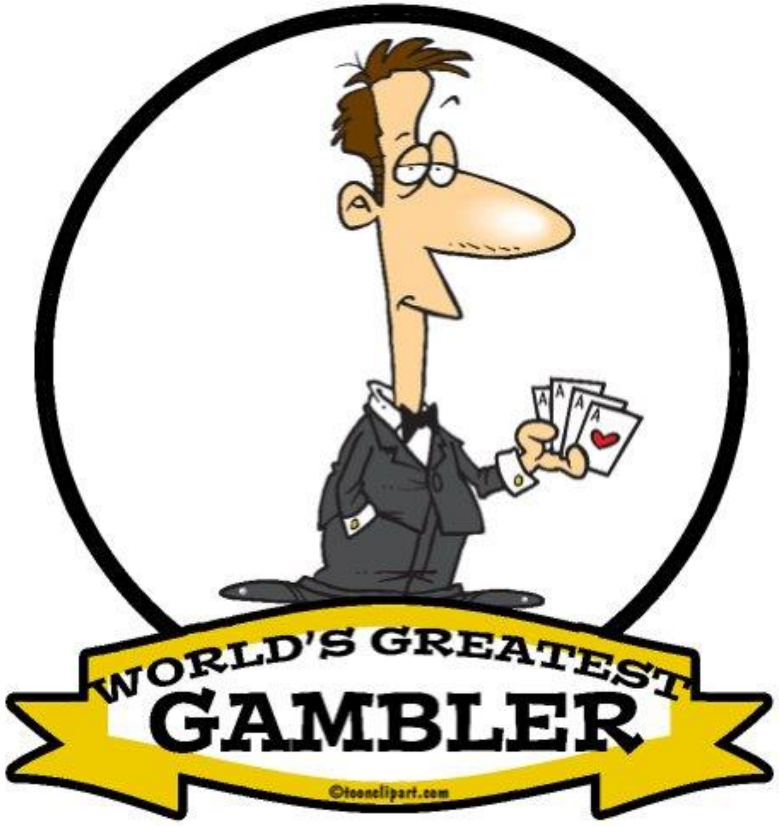
- The game is fair in the sense that the expected win/loss her game is zero.

- The gambler comes up with the following system:
  - He/she initially bets $1.
  - If this first bet loses, he/she bets $2 on the second bet. If this loses, he/she bets $4 on the third bet. If this loses, he/she bets $8 on the fourth bet etc.
- The gambler believes this to be flawless, since each "cycle" always ends with $1 profit.

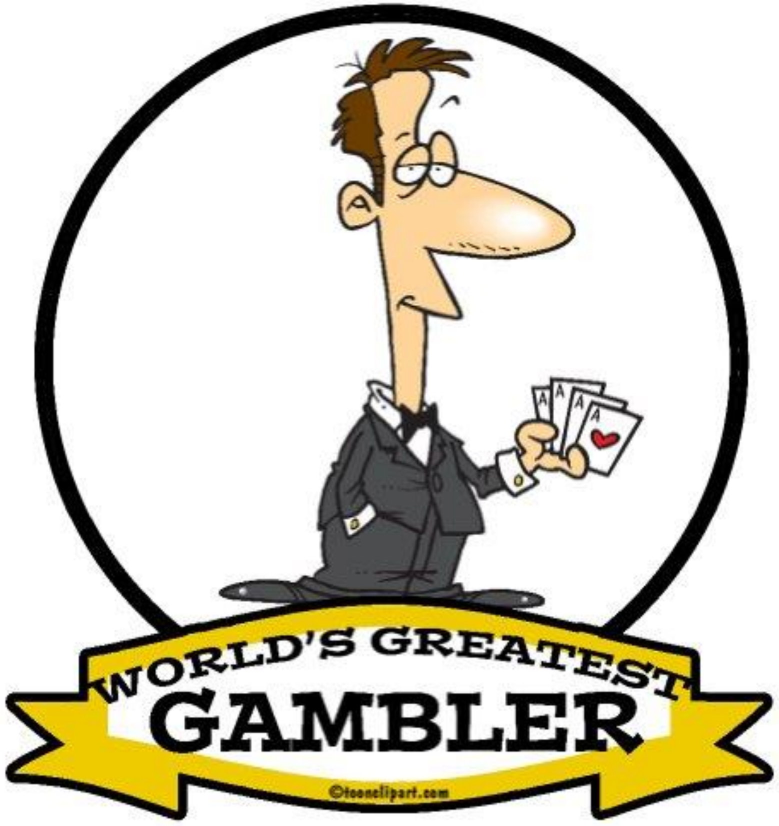
St Petersburg Paradox

- Despite not being biased either for or against the gambler, his/her probability of leaving a cycle in profit is $\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^4+\ldots=1$

- How has the gambler invented a system seeming to give guaranteed profits when the game is fair?



St Petersburg Paradox



- How much money does the gambler need before the first win?

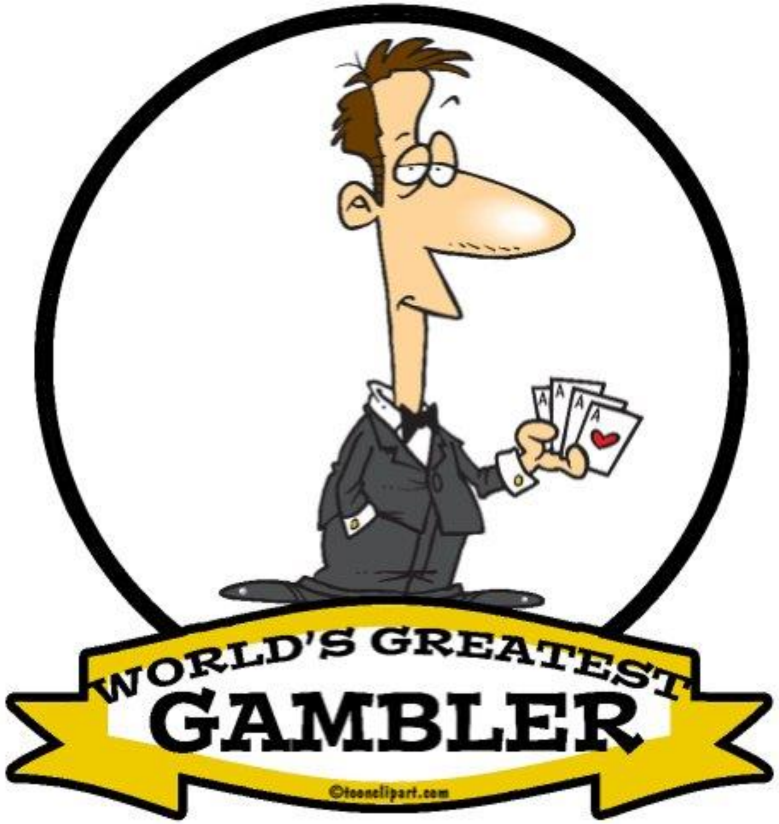
- With probability 0.5, he/she gambles a total of \$1. With probability 0.25, he/she gambles another \$2. With probability 0.125, he/she gambles another \$4 etc.

- The expected total stake of the first winning bet is therefore

$$\left(\$1\times\frac{1}{2}\right)+\left(\$2\times\frac{1}{4}\right)+\left(\$4\times\frac{1}{8}\right)+\left(\$8\times\frac{1}{16}\right)+\left(\$16\times\frac{1}{32}\right)+\ldots \quad \to \infty$$

- In other words, to ensure a \$1 profit, the gambler need to have infinite wealth available.

St Petersburg Paradox



- The St. Peterburg Paradox is sometimes also called the Gambler's Ruin.

- In reality, however, it's maybe not such a bad strategy. For example, if the gambler enters the game with $2047 and follows his strategy and exits the game upon his first win. His probability of leaving in ($1) profit is therefore

- $= 1 - \frac{1}{2^{11}} \approx 99.95\%$ (i.e. certain, unless the coin lands tails 11 times in a row).

- However… on average one gambler in every 2000 (0.05%) that follows this strategy will lose over two thousand dollars…in pursuit of a one dollar profit.

Properties of Expectations

- Consider a random variable with expectation $E(X)=\sum(k\times P(X=k))$.

- Let another random variable be $Y=2X$. Its expectation is therefore

$$E(Y)=\sum(2k\times P(Y=2k)) = \sum(2k\times P(X=k)) = 2\sum(k\times P(X=k)) = 2E(X)$$

- In general, for any constant C, we have $E(CX)=CE(X)$.

- Also, the expectation of a constant is simply the value of that constant e.g. If $Z=10$

$$E(Z)=\sum(k\times P(Z=k))=10P(Z=10)=10$$

Properties of Expectations

- Consider two random variables X and Y. The expectation of $X+Y$ is therefore

$$E(X+Y)=\sum(k\times P(X+Y=k)) \ =\sum(k-j)\times P(X=k-j))+\sum j\times P(Y=j))$$

- In general, for any two random variables $E(X+Y)=E(X)+E(Y)$.

- Again, this is reasonably intuitive. If, for example you rolled one regular fair six-sided die and also flipped one fair coin and counted how many times (0 or 1) it landed Tails, then your expected total of the coin flip and the die roll would be 3.5 plus 0.5 = 4.

- Note that $E(X+Y)=E(X)+E(Y)$, even if X and Y are not independent.

Properties of Expectations

- Consider rolling one regular fair six-sided die and recording *X*, the number shown on the die and *Y*, the number of even numbers (0 or 1) shown on the roll.
- The probability mass functions are therefore

$$P(X=k)=\begin{cases}1/6 & k=1\\ 1/6 & k=2\\ 1/6 & k=3\\ 1/6 & k=4\\ 1/6 & k=5\\ 1/6 & k=6\\ 0 & \text{otherwise}\end{cases}$$

- This gives $E(X)=3.5$ and $E(Y)=0.5$.

$$P(Y=k)=\begin{cases}0.5 & k=0\\ 0.5 & k=1\\ 0 & \text{otheriwse}\end{cases}$$

Properties of Expectations

- Consider rolling one regular fair six-sided die and recording X, the number shown on the die and Y, the number of even numbers (0 or 1) shown on the roll.

- Working out the probability mass function of $X \times Y$ however, gives

$$P(X \times Y = k) = \begin{cases} 1/2 & k = 0 \\ 1/6 & k = 2 \\ 1/6 & k = 4 \\ 1/6 & k = 6 \\ 0 & \text{otherwise} \end{cases} \quad \text{and hence} \quad E(X \times Y) = \left(0 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = 2$$

Properties of Expectations

- Consider rolling one regular fair six-sided die and recording X, the number shown on the die and Y, the number of even numbers (0 or 1) shown on the roll.

- We note that here $E(X) = 3.5$ and $E(Y) = 0.5$ but $E(X \times Y) = 2$.

- In general, $E(X) \times E(Y) \neq E(X \times Y)$ (although this is true if X and Y are independent.)

Expectation of Discrete Random Variables

- The expectation alone doesn't really give us the complete picture about a random variable.

- For example, consider being offered the chance to play one of three games.
  - Game 1 involves no randomness, and guarantees a $10 profit.
  - Game 2 involves either losing $10 or winning $30, both with probability 0.5.
  - Game 3 involves having a 1 in a billion (0.000000001) chance of winning $10,000,000,000, otherwise winning nothing.

- Each of these has the same expectation, but is very different to the other two games.

Variance of Discrete Random Variables

- To characterise the difference between the (very different) variables shown on the previous slide, we introduce the concept of variance.

- The **variance** of a random variable X is defined as $Var(X) = E((X - \mu)^2)$.

- A larger variance means that outcomes further from the expectation are more likely.

- Note that the variance cannot be negative and variance of zero implies that the system has no unpredictability i.e. is constant.

Easier Calculation of Variance

- By definition, $Var(X) = E((X-\mu)^2) \; = E(X^2 - 2\mu X + \mu^2)$

- This can then be split to give $Var(X) = E(X^2) - 2\mu E(X) + \mu^2$, since μ is simply a constant.

- Again, by definition $E(X) = \mu$, so $Var(X) = E(X^2) - 2\mu(\mu) + \mu^2$.

- Hence, we have $Var(X) = E(X^2) - \mu^2$ or $Var(X) = E(X^2) - E(X)^2$.

- In other words, we can calculate the variance by calculating the expected squared value of the variable, minus the square of the expectation of the variable.

Calculation of Variance: Example

- Again, let X be the number shown by one roll of a regular fair six-sided die.
- We have already seen that $E(X) = \left(\frac{1}{6}\times 1\right) + \left(\frac{1}{6}\times 2\right) + \left(\frac{1}{6}\times 3\right) + \left(\frac{1}{6}\times 4\right) + \left(\frac{1}{6}\times 5\right) + \left(\frac{1}{6}\times 6\right) = 3.5$
- The probability mass function of X^2 is $P(X^2 = k) = \begin{cases} 1/6 & k = 1 \\ 1/6 & k = 4 \\ 1/6 & k = 9 \\ 1/6 & k = 16 \\ 1/6 & k = 25 \\ 1/6 & k = 36 \\ 0 & \text{otherwise} \end{cases}$

Calculation of Variance: Example

- From this, we can obtain $E(X^2)=\left(\frac{1}{6}\times 1\right)+\left(\frac{1}{6}\times 4\right)+\left(\frac{1}{6}\times 9\right)+\left(\frac{1}{6}\times 16\right)+\left(\frac{1}{6}\times 25\right)+\left(\frac{1}{6}\times 36\right)=\frac{91}{6}$
- This in turn gives $Var(X)=\frac{91}{6}-\left(\frac{7}{2}\right)^2=\frac{35}{12}$