University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161)
Class Test
SOLUTIONS

Question 1.

a)

i) $\{1,2,\ldots,48,49\}$.

ii) $\{1,2,3,\ldots\}$.

iii) $\{0,1\}$.

iv) $\{0,1,2,3,4,5\}$.

v) $\{0,1,2,\ldots,9,10\}$.

vi) $\{10,11,12,\ldots\}$

b)

i) None of the given variables. Draws are not independent.

ii) $Geo\left(\frac{1}{3}\right)$.

iii) $Bern(0.3)$.

iv) None of the given variables. Individuals' surnames are not independent.

v) $Bin(10,0.25)$.

vi) None of the given variables (would be a negative binomial.)

Question 2.

a)

i) $P(X \cap Y) = P(X) + P(Y) - P(X \cup Y) = 0.15$;

ii) $P(Y^c) = 1 - P(Y) = 0.5$;

iii) $P(X|Y) = \dfrac{P(X \cap Y)}{P(Y)} = \dfrac{0.15}{0.5} = 0.3$;

iv) $P(Y|X) = \dfrac{P(X \cap Y)}{P(X)} = \dfrac{0.15}{0.4} = 0.375$;

v) $P(X^c \cap Y^c) = 1 - P(X \cup Y) = 0.25$;

vi) $P(Y|X^c \cap Y^c) = 0$.

b)

i) $P(Z \mid X) = \dfrac{P(Z \cap X)}{P(X)} = \dfrac{0.15}{0.4} = 0.375$;

ii) Because $Z \subseteq X$, we have that $P(X \cap Z) = P(Z)$ hence

$$P(X \mid Z) = \frac{P(X \cap Z)}{P(Z)} = \frac{P(Z)}{P(Z)} = 1;$$

iii) $P(Z^c) = 1 - P(Z) = 0.85$.

(5 Marks)

Question 3.

i)

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\pi}^{\pi} \frac{4x + x\sin(x)}{8\pi}\,dx$$

$$= \left[\frac{2x^2 + \sin(x) - x\cos(x)}{8\pi}\right]_{-\pi}^{\pi} = \left[\frac{2\pi}{8\pi}\right] = \frac{1}{4}$$

ii) $P\left(x > \dfrac{\pi}{2} \middle| x > 0\right) = 0.5$ since, restricted to only $x > 0$, the function is symmetrical about the line $x = \dfrac{\pi}{2}$.

This answer could also be obtained by integration, but the verbal explanation is probably simpler. Either is acceptable.

(4 Marks)

Question 4.

Many possible answers. For *A* and *B* to be independent, there are three options.
Either:

- At least one of *A* and *B* is empty. (In this case $P(A)P(B) = P(A \cap B) = 0$ hence independent.)
- At least one of *A* and *B* is Ω and the other is non-empty (In this case $P(A)P(\Omega) = P(A) \times 1 = P(A \cap \Omega) = P(A)$ (assuming *B* is the whole sample space) hence independent)
- Either *A* contains three elements and *B* contains two elements or vice versa, and one element is in both sets.

For *A* and *C* to be mutually exclusive, they must have no elements in common.

Write down three possible events *A*, *B* and *C* such that *A* and *B* are independent and *A* and *C* are mutually exclusive.

One possible answer therefore is that $A = \{1,2,3\}$, $B = \{3,4\}$, $C = \{6\}$

(3 Marks)

Question 5.

a)

i) $E(Y) = 0(0.1) + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 8\left(\frac{1}{8}\right) - 4(0.025) = 2.9$;

ii) $E(Y^2) = 0^2(0.1) + 2^2\left(\frac{1}{2}\right) + 4^2\left(\frac{1}{4}\right) + 8^2\left(\frac{1}{8}\right) + (-4)^2(0.025) = 14.4$.

(2 Marks)

b)

i) $Var(Y) = E(Y^2) - E(Y)^2 = 14.4 - 2.9^2 = 5.99$;

ii) $P(Y > 0) = P(Y = 2) + P(Y = 4) + P(Y = 8) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$;

iii) $P(Y^2 > 0) = 1 - P(Y = 0) = 0.9$;

iv) $P(Y = 4 \mid Y^2 = 16) = \frac{P(Y = 4)}{P(Y = 4) + P(Y = -4)} = \frac{0.25}{0.25 + 0.025} = \frac{10}{11}$;

v) $P(Y = 6 \mid Y^2 = 36)$ is not defined, since $P(Y^2 = 36) = 0$.

(6 Marks)

Question 6.

A player rolls a regular fair six-sided die repeatedly and notes the number shown each time. He/she also draws one of three cards with each card equally likely to be selected.

If the player selects Card A, he/she scores zero points every time the die shows an even number and one point otherwise.

If the player selects Card B, he/she scores zero points every time the die shows a number divisible by 3 and one point otherwise.

If the player selects Card C, he/she scores zero points every time the die shows a number divisible by 6 and one point otherwise.

Let S be the number of points scored and let A, B and C be the events that the card chosen was A, B or C respectively.

a) The player selects a card once, then rolls the die three times.

i) $$P(S=0)=P(S=0\,|\,A)P(A)+P(S=0\,|\,B)P(B)+P(S=0\,|\,C)P(C)$$

$$=\left(\frac{3^3}{6^3}\right)\left(\frac{1}{3}\right)+\left(\frac{2^3}{6^3}\right)\left(\frac{1}{3}\right)+\left(\frac{1^3}{6^3}\right)\left(\frac{1}{3}\right)=\frac{1}{18}$$

ii) $$P(B\,|\,S=0)=\frac{P(B\cap S=0)}{P(S=0)}=\frac{\left(\frac{2^3}{6^3}\right)\left(\frac{1}{3}\right)}{\frac{1}{18}}=\frac{2}{9}.$$

(4 Marks)

b) The player selects a card once, then rolls the die three times and is told the number of points he/she has scored.

The only possible sequences of three outcomes for which the conditional probability of having selected each card given the observed score is the same as the unconditional probability without knowing the scores are either rolling three 2s in a row or three 4s in a row, which would score 3 points, regardless of which card has been selected.

(3 Marks)