SEAT NUMBER:

STUDENT NUMBER:

SURNAME:
(FAMILY NAME)

OTHER NAMES:

This paper and all materials issued must be returned at the end of the examination.
They are not to be removed from the exam centre.

Examination Conditions:

It is your responsibility to fill out and complete your details in the space provided on all the examination material provided to you. Use the time before your examination to do so as you will not be allowed any extra time once the exam has ended.

You are **not** permitted to have on your desk or on your person any unauthorised material. This includes but not limited to:

- Mobile phones
- Smart watches and bands
- Electronic devices
- Draft paper (unless provided)
- Textbooks (unless specified)
- Notes (unless specified)

You are **not** permitted to obtain assistance by improper means or ask for help from or give help to any other person.

If you wish to **leave and be re-admitted** (including to use the toilet), you have to wait until **90 mins** has elapsed.

If you wish to **leave the exam room permanently**, you have to wait until **60 mins** has elapsed.

You are **not** permitted to leave your seat (including to use the toilet) during the final 15 mins.

During the examination **you must first seek permission** (by raising your hand) from a supervisor before:

- Leaving early
- Using the toilet
- Accessing your bag

Misconduct action will be taken against you if you infringe university rules.

Declaration: I declare that I have read the advice above on examination conditions and listened to the examination supervisor's instructions for this exam. In addition, I am aware of the university's rules regarding misconduct during examinations. I am not in possession of, nor do I have access to, any unauthorised material during this examination. I agree to be bound by the university's rules, codes of conduct, and other policies relating to examinations.

Signature:

Date:

37161 Probability and Random Variables

Time Allowed: 2 hours and 10 mins

Includes 10 minutes of reading time.

Reading time is for reading only. You are not permitted to write, calculate or mark your paper in any way during reading time.

This is a Closed Book exam

Please refer to the permitted materials below:

Permitted materials for this exam:

- Calculators (non-programmable only)
- Drawing instruments
  i.e. Rulers, Set Squares and Compasses

Materials provided for this exam:

- This examination paper
- Five (5) answer booklets (5 pages)

Do not open your exam paper until instructed.

Rough work space

Do **not** write your answers on this page.

Question 1 ((1+1+1+1+1+1+1+1+1+1+1+1) + (1+1+1+1+1+1+1+1) = 20 marks)
Start a new answer booklet

a) Let $X \sim Bin(8, 0.25)$ and $Y \sim Geo(0.3)$ be independent random variables.
Y has probability mass function

$$P(Y = k) = \begin{cases} 0.3(0.7)^{k-1} & k \in \{1,2,3,...\} \\ 0 & \text{otherwise} \end{cases}$$

i) Write down the probability mass function for X.

That is, write down $P(X = k)$ for all real values of k.

Calculate:

ii) $E(X)$;

iii) $P(X = 3)$;

iv) $P(X > 6)$;

v) $P(X \neq 6)$;

vi) $P(X = E(X))$;

vii) $P(Y = E(Y))$;

viii) $P(X > 7.3 \mid X > 6)$;

ix) $P(X > 7.3 \mid X > 7)$;

x) $P\left(\dfrac{X^2}{Y^2} = 9\right)$;

xi) $P(X = 0 \mid XY = 0)$.

xii) Show that Y has the no-memory property.

That is, show that for any positive $n \in \{1,2,3,...\}$,
$P(Y > k + n \mid Y > n) = P(Y > k)$ for all $k \in \{1,2,3,...\}$

b) Let A and B be events in a sample space Ω such that

$P(A) = 0.5,\ P(A \cap B) = 0.15$ and $P(A \cup B) = 0.7$.

Calculate:

i) $P(A^c)$;

ii) $P(B)$;

iii) $P(A \mid B)$;

iv) $P(B \mid A)$;

v) $P(B \mid A^C \cap B^c)$;

vi) $P((A \cup B)^c)$.

vii) Are A and B independent? Justify your answer.

Let the event C in Ω be independent of both A and B such that $P(A \cap C) = 0.1$.

viii) Calculate $P(B \cap C)$.

Over…

Question 2 ((2+2+2+2+2+4) + (2+2+2) = 20 marks)

Start a new answer booklet

a) A geometric random variable $Y \sim Geo(p)$ has probability mass function

$$P(Y=k)=\begin{cases} p(1-p)^{k-1} & k\in\{1,2,3,...\} \\ 0 & \text{otherwise} \end{cases}.$$

i) Show that the generating function of Y, $g_Y(z)=E(z^Y)=\dfrac{pz}{1-(1-p)z}$.

The random variable V describes how many independent $Bern(p)$ variables must be counted until the nth $(n\in\mathbb{Z}^+)$ 1 is seen.
This describes a negative binomial variable, $V \sim NegBin(n,p)$.

(For example, if the sequence of $Bern(p)$ variables were 0,0,1,0,1,… then the corresponding $NegBin(2,p)$ variable would take the value 5 since the 2nd 1 is seen on the 5th $Bern(p)$ variable)

ii) What is the range of V? Justify your answer.

iii) Calculate the generating function of V.

An exponential random variable $W \sim \exp(\lambda)$ has probability density function

$$f(w)=\begin{cases} \lambda e^{-\lambda w} & w\in[0,\infty) \\ 0 & \text{otherwise} \end{cases}.$$

iv) Show that the generating function of W is $g_W(z)=E(z^W)=\dfrac{\lambda}{\lambda-\log(z)}$.

v) By differentiating the generating function, show that $E(W)=\dfrac{1}{\lambda}$.

vi) Show that, if $W_1,W_2,W_3,...$ are independent random variables such that each $W_i \sim \exp(\lambda)$ and if $Y \sim Geo(p)$ and $S=\sum_{i=1}^{Y} W_i$, then S is exponentially distributed. State its rate parameter.

Hint: You may use without proof the result that, if $S=\sum_{i=1}^{Y} W_i$, $g_S(z)=g_Y(g_{W_i}(z))$.

Over…

Question 2 (Continued)

b) Two types of a disease affect a large population. 1% of the population has Type A of the disease, 2% of the population has Type B of the disease and 97% of the population does not have either form of the disease.

A test exists to detect the disease in patients.

If a patient has Type A, the test correctly classifies the patient with probability 0.8, and misclassifies him/her as Type B or disease-free both with probability 0.1

If a patient has Type B, the test correctly classifies the patient with probability 0.9, and misclassifies him/her as disease-free with probability 0.1

If a patient is disease-free, the test correctly classifies the patient with probability 0.7 and misclassifies him/her as either Type A or Type B each with probability 0.15.

One individual is selected at random from the population and tested. Assume all people in the population are equally likely to be chosen.

i) Calculate the probability that the selected individual is misclassified by the test. That is, find the probability that the patient's true health status and the result of his/her test are not the same.

ii) Show that the probability that the patient is told that he/she does not have the disease is 68.2%.

iii) Calculate, given a patient is told he/she is disease-free, the probability that this classification is correct.

Over…

Question 3 ((1+2+1+2+2+2+2) + (2+1+4+1) = 20 marks)

Start a new answer booklet

a) A 24 hour maternity ward models the births of babies by independent Poisson processes such that, on average, 6 boys and 6 girls are born on the ward each day. 90% of births are classified as happening without complication and 10% are complicated. Whether or not complications arise is assumed to be independent of the sex of the baby.

Calculate the probability that:

i) During one whole day, exactly four girls are born;

ii) During one whole day, exactly four babies are born and all are girls;

iii) During a 12 hour shift, no babies are born from births with complications;

iv) From a selection of 10 births selected uniformly at random, all 10 have no complications;

v) During one whole day, at least 11 babies are born given that 6 boys are born without complications and 9 girls are born, of which two were complicated births.

Find the distributions of the following variables:

vi) The number of boys born during one hour;

vii) The time (in days) between successive births with complications.

b) One player plays a game which involves rolling a regular fair six-sided die repeatedly. Each time the die lands on a 1 or a 3, the player subtracts one point from his/her total. Each time the die lands on a 5, the player adds one point to his/her total and each time it lands on an even number, his/her points total does not change.

The player starts with 15 points and rolls the die repeatedly until he/she either first reaches 20 points (wins) or first reaches 0 points (loses.)

Let W_k be the probability that the player wins the game given that he/she has k points at a given time.

i) Show that W_k satisfies $3W_k = W_{k+1} + 2W_{k-1}$.

Clearly explain your calculation in your own words.

ii) Write down the boundary conditions. That is, write down the values of W_0 and W_{20}

.

iii) Solve the difference equation to find the value of W_k for any $k \in \{0,1,2,...,20\}$.

iv) Hence or otherwise, show that $W_{15} \approx 3.12\%$

Over…

Question 4 ((1+2+1+3+2+1+1+1) + (2+2+2+2)) = 20 marks)

Start a new answer booklet

a) Let X be a discrete random variable with probability mass function

$$P(X=k)=\begin{cases}0.05 & k=0\\ \dfrac{1}{k} & k\in\{3,4,5,6\}\\ 0 & \text{otherwise}\end{cases}$$

and let Y be a continuous random variable with cumulative density function

$$F(y)=P(Y\le y)=\begin{cases}0 & y<0\\ \dfrac{\sin(y)}{2}+\dfrac{y}{\pi} & y\in\left[0,\dfrac{\pi}{2}\right].\\ 1 & y>\dfrac{\pi}{2}\end{cases}$$

i) Verify that $\sum_k P(X=k)=1$.

Calculate:

ii) $f(y)$, the probability density function of Y.

iii) $E(X)$;

iv) $E(Y)$;

v) $E\left(\dfrac{1}{X}+X+3\right)$;

vi) $P(Y=2)$;

vii) $P(X=5\,|\,Y\neq 1)$;

viii) $P(X>Y)$.

b) Let Q and R be discrete random variables and let S and T be continuous random variables.

Write down a possible probability mass function for Q and R such that:

i) $P(0\le Q\le 2\,|\,0\le Q\le 3)=0.5$;

ii) $E(R)=1.7$ and $Var(R)=0$.

Write down a possible probability density function for S and T such that:

iii) $P(0\le S\le 2\,|\,0\le S\le 3)=0.5$;

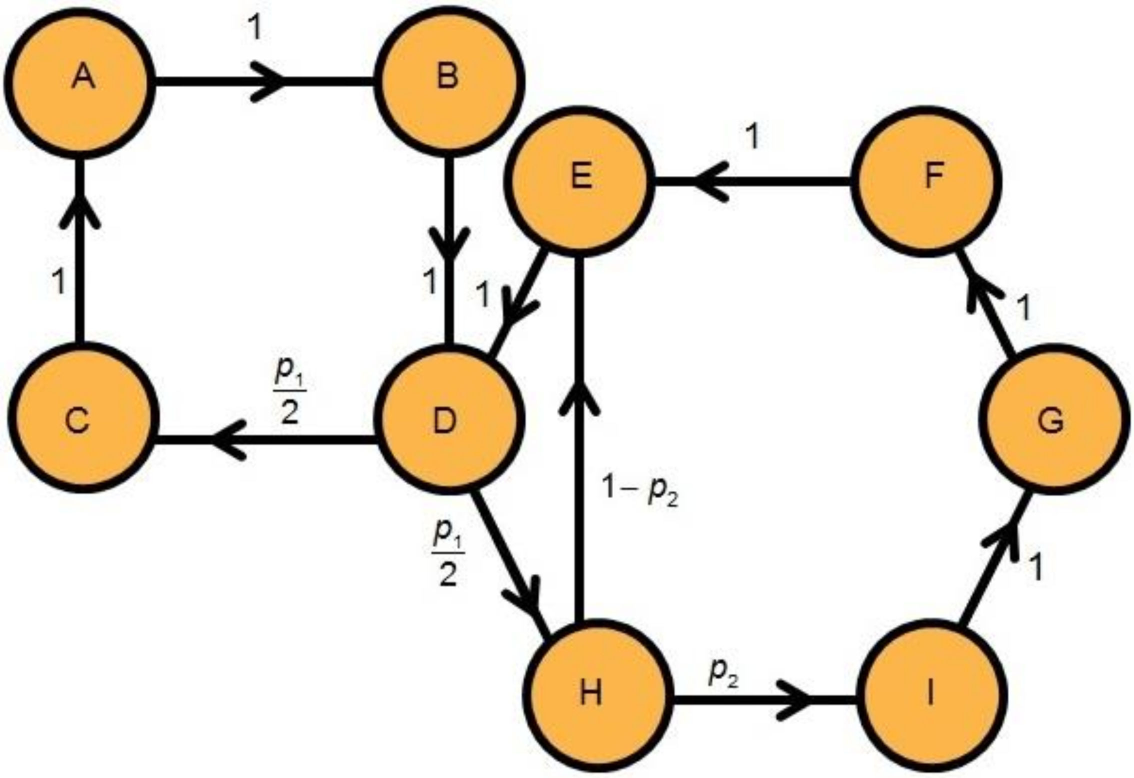
iv) $E(T)=0$ and $Var(T)>20$.

Over…

Question 5 ((2+2+2+2+2+2) + (3+2+3) = 20 marks)

Start a new answer booklet

a) Let $X_0, X_1, X_2, \ldots$ be a Markov Chain which is represented by the state diagram



i) Find the transition matrix for this chain.

An absorbing state is one such that, if the system ever enters that state, the probability that it is ever in a different state is zero.

A transient state is one such that, if the system is ever in that state and moves to another state, the probability that it eventually returns is less than one.

A state *i* is periodic with period $d > 1$ if $P(X_{n+k} = i \mid X_n = i) = 0$ for all *n* unless *k* is divisible by *d*.

ii) Find the period of each state if $p_1 = 1, p_2 = 0$.

iii) Find the period of each state if $p_1 = 1, p_2 = 1$.

iv) Find the period of each state if $p_1 < 1$.

v) Which is the only state which can be absorbing?
State the values of p_1 and p_2 for which this state is absorbing.

vi) Which states can be transient?
State the values of p_1 and p_2 for which each of these states is transient.

Over…

Question 5 (Continued)

b) Let $Y_0, Y_1, Y_2, \ldots$ be a Markov Chain with transition matrix $P = \begin{pmatrix} 0.8 & 0.1 & 0.1 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0.05 & 0.05 & 0.9 \end{pmatrix}$.

i) Draw the state diagram for this chain.

ii) Calculate the 2-step transition probability $P(Y_{n+2} = 2 | Y_n = 3)$.

iii) Calculate the equilibrium distribution for this chain.

Hint: You might find some of the following information useful.

$$\begin{pmatrix} 0.8 & 0.3 & 0.3 & 0 \\ 0.1 & 0.4 & 0.3 & 0.05 \\ 0.1 & 0.3 & 0.4 & 0.05 \\ 0 & 0 & 0 & 0.9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.1 \\ -0.1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0.8 & 0.3 & 0.3 & 0 \\ 0.1 & 0.4 & 0.3 & 0.05 \\ 0.1 & 0.3 & 0.4 & 0.05 \\ 0 & 0 & 0 & 0.9 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.5 \\ -0.5 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0.8 & 0.3 & 0.3 & 0 \\ 0.1 & 0.4 & 0.3 & 0.05 \\ 0.1 & 0.3 & 0.4 & 0.05 \\ 0 & 0 & 0 & 0.9 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.8 & 0.3 & 0.3 & 0 \\ 0.1 & 0.4 & 0.3 & 0.05 \\ 0.1 & 0.3 & 0.4 & 0.05 \\ 0 & 0 & 0 & 0.9 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 5.4 \\ 0.9 \\ 0.9 \\ -7.2 \end{pmatrix}$$

End of Paper